On Baum Connes conjecture

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Basic generalisation of a locally compact Hausdorff space is a C*-algebra. The idea is to look at the functor

\[ X \mapsto C_0(X) \]

and replace \( C_0(X) \) by a non-commutative C*-algebra. A C*-algebra is a norm closed subalgebra of \( B(H) \) (bounded operators on a Hilbert space \( H \)) closed under taking adjoints \( a \to a^\ast \). First examples

1. \( M_n(\mathbb{C}); \; H = \mathbb{C}^n \),
2. \( C_0(X); \; h = L^2(X, \mu) \) where \( \mu \) is any positive Radon measure nonvanishing on any open subset of \( X \) and \( f \in C_0(X) \) acts by multiplication

\[ L^2(X) \ni \xi \to f\xi \in L^2(X). \]

In fact, any abelian C*-algebra is of this form.

3. \( K(H) \) the algebra of all compact operators on \( H \).
The basic norm identity is

\[ ||a^* a|| = ||a||^2. \]

C*-algebras form a category, with

\[ \text{Mor}_{C^*}(A, B) = \{ \phi : A \to B \mid \phi \text{ is a } *\text{-homomorphism} \}. \]

The basic C*-identity implies a sensible notion of positivity, and in particular, every *-homomorphism is automatically continuous. What distinguishes a C*-algebra from complex numbers is the fact that the unit ball is not round.
We can always add some extra structure, f. ex. a $G$-action

$$\alpha : G \rightarrow Aut(A)$$

by $\ast$-automorphisms, where $G$ is a (second countable) locally compact group and $\alpha$ is a pointwise continuous homomorphism. In this case

$$\text{Mor}^G_C(A, B)$$

consists of $\ast$-homomorphisms preserving group action.
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**Topology**

The category of Abelian $G$-$\ast$-algebras coincides with the category of pointed compact Hausdorff $G$-spaces.
Definition

A non-commutative homology theory is a functor on a category of (separable) $C^*$-algebras (with extra structure) that is

- $C^*$-stable (Morita invariant)
- split-exact
- homotopy invariant
- has Puppe exact sequence for mapping cones

Example: $\Gamma = \mathbb{Z}$
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Example

$K$-theory is a non-commutative homology theory for $C^*$-algebras. It maps separable $C^*$-algebras to the category $\text{Ab}_{\mathbb{Z}/2}^c$ of $\mathbb{Z}/2$-graded countable Abelian groups.
Example

$KK^G$ is a (bivariant) non-commutative homology theory for $C^*$-algebras with a $G$-action.

Cycles in $KK^G(A, B)$

- $\mathcal{H}_B$ is a right Hilbert $B$-module;
- $\varphi: A \to B(\mathcal{H}_B)$ is a $*$-representation;
- $F \in B(\mathcal{H}_B)$;
- $\varphi(a)(F^2 - 1)$, $\varphi(a)(F - F^*)$, and $[\varphi(a), F]$ are compact for all $a \in A$;
- in the even case, $\gamma$ is a $\mathbb{Z}/2$-grading on $\mathcal{H}_B$;
- $\mathcal{H}_B$ carries a representation $U$ of $G$ which implements action of $G$ and commutes with $F$ up to compacts.

A cycle is trivial, if all the "compacts" above vanish, and two cycles are equivalent, if they are homotopic after adding trivial cycles.
Some properties of $KK^G$

1. The classes in $KK^G_1(A, B)$ are given by semisplit extensions: $0 \to B \otimes K \to E \to A \to 0$

2. Kasparov product
   
   $KK^G_i(A, B) \times KK^G_j(B, C) \to KK^G_{i+j}(A, C)$

3. Excision. Given a semisplit short exact sequence
   
   $0 \to I \to A \to A/I \to 0$, there exists an associated six term exact sequence

   $$
   KK^G_0(A/I, B) \to KK^G_0(A, B) \to KK^G_0(I, B)
   $$

   and similarly in the second variable.

4. For $G$ compact group
   
   - $KK^*_G(\mathbb{C}, A) = K^*_G(A) = K_*(A \rtimes G)$ - equivariant $K$-theory
   - $KK^*_G(\mathbb{C}, \mathbb{C}) = R_G$ - the representation ring of $G$. 

Suppose that $G = \mathbb{Z}$. Then

The cycles are given as follows

- An even representation of $\mathbb{Z}$ on a Hilbert space $H = H^+ \oplus H^-$ (hence a pair of unitary operators $U^+ \oplus U^-$),

- A Fredholm operator $F : H^+ \to H^-$ which intertwines $U^+$ with $U^-$ modulo compacts.

Then the class of $(U, F)$ gives

$$\text{Index}(F) = \dim \ker F - \dim \coker F \in \mathbb{Z}.$$ 

**Theorem (BC for $\mathbb{Z}$)**

$$KK^\mathbb{Z}_0 (\mathbb{C}, \mathbb{C}) \ni F \to \text{Index}(F) \in \mathbb{Z}$$

is an isomorphism.
The Kasparov product

\[ KK^*_G(C, B) \times KK^*_G(B, C) \to KK^*_G(C, C) \]

has an explicit description as follows.

Given class \([D] \in KK^*_G(B, C)\), represent it by a semisplit extension

\[ 0 \to C \otimes K \to E \to B \to 0. \]

Then the pairing

\[ \cap [D] : K^*_G(B) \to K^*_G(C) \]

coincides with the boundary map \(\delta\) in the six-term exact sequence

\[ K^0_G(C) \to K^0_G(E) \to K^0_G(B) \]
\[ \to K^1_G(B) \leftarrow KK^1_G(E) \leftarrow K^1_G(C) \]

\[ \delta \]
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The universality of Kasparov theory

Theorem (Joachim Cuntz and Nigel Higson)

**Bivariant KK-theory is the universal C*-stable, split-exact functor on the category of separable C*-algebras.**

That is, a functor from the category of separable C*-algebras to some additive category factors through KK if and only if it is C*-stable and split-exact, and this factorisation is unique if it exists.
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The universality of Kasparov theory

**Theorem (Joachim Cuntz and Nigel Higson)**

*Bivariant KK-theory is the universal $C^*$-stable, split-exact functor on the category of separable $C^*$-algebras. That is, a functor from the category of separable $C^*$-algebras to some additive category factors through $KK$ if and only if it is $C^*$-stable and split-exact, and this factorisation is unique if it exists.*

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**Corollary**

$C^*$-stability and split-exactness

$\longrightarrow$ homotopy invariance, Bott periodicity, Connes–Thom Isomorphism, ...
Let $KK^G$ be the category of $G$-$C^*$-algebras (separable) with morphisms given by $KK_0^G$ (the composition of morphisms is given by Kasparov product.

**Theorem**

The following gives $KK^G$ triangulated structure

1. **Shift** $A \to SA = C_0(\mathbb{R}, A)$
2. **Exact triangles**

\[
\begin{array}{ccc}
A & \xrightarrow{} & B \\
\downarrow & & \downarrow \\
E & \xleftarrow{} & A \\
\end{array}
\]

are given by semisplit extensions

\[0 \to SB \to E \to A \to 0\]

**Definition**

Set $\alpha : A \to C(G, A)$ to be the $*$-homomorphism $\alpha(a)(g) = g^{-1}(a)$ The reduced crossed product, 

$$A \rtimes_{\text{red}} G$$

is the $C^*$-algebra on $A \otimes L^2(G)$ generated by (products of elements in) $\alpha(A)$ and the regular representation of $G$. 

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**Definition**

Basic object of study is the functor 

\[ KK^G \ni A \Rightarrow F(A) = K_*(A \rtimes_{\text{red}} G) \in \text{Ab}^{\mathbb{Z}/2\mathbb{Z}}. \]

This is essentially the functor which describes harmonic analysis for group actions. It is homotopy invariant, but not excisive. Basic reason is the fact the functor \( A \Rightarrow A \rtimes_{\text{red}} G \) is in general not exact.

"Assembly"

Given a homotopy functor \( F \), construct a homology (excisive) functor \( \mathbb{L}F \) and natural transformation \( \mathbb{L}F \Rightarrow F \), universal for this situation

We will use the triangulated structure of \( KK^G \).
Let $\mathcal{I}$ be an ideal in $KK^G$ given by

\[ \{ j \mid j = 0 \text{ in } KK^H, \text{ for every compact subgroup } H \subset G \} \]

There is the corresponding projective class $\mathcal{P}$ in $KK^G$, consisting of the collection of algebras $P$ satisfying

\[ \mathcal{I}(A, B) \circ KK^G(P, A) = 0 \]

for all $A$, $B$. 

Example: $\Gamma = \mathbb{Z}$.
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\[
\begin{array}{l}
\text{Example} \\
\text{1. } \mathcal{I} = KK^\Gamma \text{ for a discrete group } \Gamma \\
\text{2. } j \in \mathcal{I} \text{ if, for all torsion subgroups } H \subset \Gamma, j = 0 \text{ in } KK^H \\
\text{3. } \mathcal{P} \text{ coincides with the usual class of proper } \Gamma\text{-algebras.}
\end{array}
\]
Theorem

There are enough projectives in $KK^G$, and, given any $A \in KK^G$, there exists a projective cover

$$P_A \in \mathcal{P}, \quad D_A \in KK^G(P_A, A)$$

universal for morphisms from $\mathcal{P}$ to $A$
**Definition**

K-homology Let $E_G$ be the universal proper action of $G$ (it exists!)

$$K_G^*(A) = \lim \{ KK_G^*(C(X), A) \mid X \subset E_G, X/G \text{ compact} \}$$

In the case when $A = C(M)$ is abelian, this is the usual equivariant K-homology of $M$.

**Theorem**

$$K^*_*(P_A \rtimes G) = K^*_G(A) \text{ and the assembly for } F \text{ is given by}$$

$$K_G^*(A) = K^*_*(P_A \rtimes G) \xrightarrow{D_A} K^*_*(A \rtimes_{red} G)$$
Baum Connes conjecture

The assembly map

\[ K_G^*(A) \to K_*(A \rtimes_{\text{red}} G) \]

is an isomorphism.
Baum Connes conjecture

The assembly map

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Status

1. True for discrete groups acting properly isometrically on Hilbert spaces
2. True for almost connected groups (Connes Kasparov conjecture)
3. True for \( \text{Sp}(n,1) \)
4. Open for \( \text{SL}(3,\mathbb{Z}) \)
5. False for "non-exact groups" (if they exist).
Corollaries of BC

1. Injectivity of assembly implies Novikov conjecture (Higher \(L\)-genera are homotopy invariant)

2. Surjectivity of assembly implies Kaplansky conjecture (for torsion free \(G\), \(C^*_{\text{red}}(G)\) has no nontrivial idempotents.)
In general, it is enough to find the "Dirac" element $D = D_\mathbb{C}$, since

$$P_A = P_\mathbb{C} \rtimes G$$

**Remark**

Since $P_\mathbb{C}$ is a projective cover, there exists an Adams type spectral sequence computing $K_*^G(A)$

$G$ has a $\gamma$-element, if $D_\mathbb{C} \in KK^G(P_\mathbb{C}, \mathbb{C})$ has a left inverse $Q$, and then $\gamma_G = QD_\mathbb{C} \in KK^G(\mathbb{C}, \mathbb{C})$. 
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All proofs of BC go via showing that $\gamma_G$ acts as identity on $K_*(\cdot \rtimes_{red} G)$
$G$ satisfies the strong Baum-Connes conjecture, if $\gamma_G = 1$. This is equivalent to saying that every object in $KK^G$ is in the localizing category generated by the subcategory of projectives.
On Baum Connes conjecture

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Non-commutative topology
C*-algebras
Homology
KK^G
Kasparov product

KK^G-category
Assembly
Baum-Connes conjecture
Categorical reformulation
Example: Γ = Z

- Γ = Z
- $\mathcal{I} = \text{Ker}: KK^\mathbb{Z} \to KK$

The $\mathcal{I}$-projective resolution of $\mathbb{C}$ has the form

\[
\begin{array}{ccccccccc}
\mathcal{K}(l^2(\mathbb{Z})) & \rightarrow & C \simeq \Sigma c_0(\mathbb{Z}) & \rightarrow & 0 \\
\pi & & \Sigma & & \\
c_0(\mathbb{Z}) & \leftarrow & 1-\sigma & \leftarrow & c_0(\mathbb{Z})
\end{array}
\]

The projective cover of $\mathbb{C} \simeq_{KK^\mathbb{Z}} \mathcal{K}(l^2(\mathbb{Z}))$ is just the mapping cone

$c_0(\mathbb{Z}) \rightarrow c_0(\mathbb{Z}) \rightarrow \Sigma C_{1-\sigma}$.

But this is just the rotated exact triangle associated to the extension

$0 \rightarrow \Sigma c_0(\mathbb{Z}) \rightarrow C_0(\mathbb{R}) \rightarrow c_0(\mathbb{Z}) \rightarrow 0$,

the $*$-homomorphism $C_0(\mathbb{R}) \rightarrow c_0(\mathbb{Z})$ given by the evaluation $f \rightarrow f|_\mathbb{Z}$. 
Conclusion

$P_{\mathbb{C}} = C_0(\mathbb{R}^2)$, $D = \overline{\partial}$, the usual Dirac operator (or rather its phase),

$$K_{\mathbb{Z}}^*(A) = K_*((A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z}) \to K_*(A \rtimes \mathbb{Z}),$$

where the assembly map is given by the product with Dirac operator.

The spectral sequence computing $K_{\mathbb{Z}}^*(A)$ becomes the six term exact sequence in $K$-theory associated to the extension

$$\Sigma(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \hookrightarrow (A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z} \twoheadrightarrow (A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z}$$

Since $(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \simeq A \otimes \mathcal{K}$, this is just the usual Pimsner-Voiculescu exact sequence.