

Shape theories of commutative and non-commutative spaces

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SYM lecture

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§0: Introduction

§1: (Approximative) Absolute (Neighborhood)
Retract and (Weakly) (Semi-)Projective
(abbreviated by (A)A(N)R and (W)(S)P)

§2: Commutative spaces
in Noncommutative Shape theory

§3: Results on Noncommutative Shape theory

Commutative shape theory

Homotopy theory is not perfect
for all compact metric space X

approximate a space X by nicer spaces
(= absolute neighborhood retracts (ANRs) X_k)

Theorem

Every X is an inverse limit of ANRs $(X_k)_k$:
$$\varprojlim (X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_k \leftarrow \cdots) \cong X$$

study approximating system $(X_k)_k$
instead of original space X
(e.g. shape equivalence)

Noncommutative shape theory

Noncommutative space = C^* -algebra
(contravariantly)

approximate an arbitrary C^* -algebra A by
nicer (= semiprojective (SP)) C^* -algebras A_k

Problem (Blackadar '85)

Is every A a direct limit of SP C^ -algebras $(A_k)_k$?*

$$\lim_{\rightarrow} (A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k \rightarrow \cdots) \cong A?$$

To solve this problem, we want lots of
examples of semiprojective C^* -algebras.

Recent progress (after '10)

- T. A. Loring and T. Shulman (§1)
- “Semiprojectivity and Asymptotic Morphisms”
by S. Eilers and T. Shulman
- D. Enders (§3)
- A. P. W. Sørensen and H. Thiel
(and D. Enders) (§2)
- S. Eilers and T. Katsura (§3)
- H. Thiel (§1)
- ...
- T. A. Loring, U. Otogonbayar, T. Shulman,
A. P. W. Sørensen, T. Katsura and ...
- ...

Commutative spaces

Definition

$\mathcal{C}pt$: the category of compact spaces

$\mathcal{C}pt_*$: the category of pointed compact spaces

Definition

$\mathcal{E}: \mathcal{C}pt \rightarrow \mathcal{C}pt_*$: Adding disjoint base points

$\mathcal{F}: \mathcal{C}pt_* \rightarrow \mathcal{C}pt$: forget base points

Identify a pointed compact space $(X, *)$

with a locally compact space $Y = X \setminus \{*\}$

$(\mathcal{C}pt_* \ni (X, *) \mapsto X \setminus \{*\} \in \mathcal{L}oc\mathcal{C}pt$ is “bijective”)

$\mathcal{E}: X \mapsto Y = X$: compact space is locally compact

$\mathcal{F}: Y \mapsto X = Y \cup \{*\}$: one-point compactification

Commutative algebra $C_0(X, *)$

Definition

For $(X, *) \in \mathfrak{Cpt}_*$

$$C_0(X, *) := \{f : X \rightarrow \mathbb{C} \mid \text{continuous, } f(*) = 0\}$$

$$C_0(X, *) = C_0(Y) \text{ for } Y = X \setminus \{*\}$$

$C_0(X, *)$: commutative \mathbb{C} -algebra

with involution $*$ and norm $\|\cdot\|_\infty$

$$f^*(x) := \overline{f(x)} \quad \text{for } f \in C_0(X, *), x \in X$$

$$\|f\|_\infty := \sup_{x \in X} |f(x)| \quad \text{for } f \in C_0(X, *)$$

C*-algebras

Definition

C*-algebra = \mathbb{C} -algebra with involution $*$
and norm $\| \cdot \|$ satisfying

Definition

A, B : C*-algebra

$\varphi: A \rightarrow B$: *-homomorphism \iff

φ : \mathbb{C} -algebra hom preserving involution $*$

Definition

$\mathbb{C}^*\text{-alg}$: Category of C*-algebras

$\mathbb{C}^*\text{-alg}_1$: Category of unital C*-algebras

Noncommutative (NC) spaces

Theorem

$\mathfrak{Cpt}_* \ni (X, *) \mapsto C_0(X, *) \in \mathfrak{C}^*\text{-alg}$
(and $\mathfrak{Cpt} \ni X \mapsto C(X) \in \mathfrak{C}^*\text{-alg}_1$): “isomorphism”
onto commutative (unital) C^* -algebras

$\mathfrak{C}^*\text{-alg}^{op}$ (resp. $\mathfrak{C}^*\text{-alg}_1^{op}$) can be called category
of NC locally compact (resp. compact) spaces

$\mathcal{E} : \mathfrak{C}^*\text{-alg}_1 \ni A \mapsto A \in \mathfrak{C}^*\text{-alg}$: forget the unit

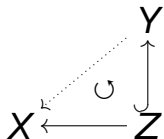
$\mathcal{F} : \mathfrak{C}^*\text{-alg} \ni A \mapsto A^+ \in \mathfrak{C}^*\text{-alg}_1$:

add the unit (= NC one-point compactification)

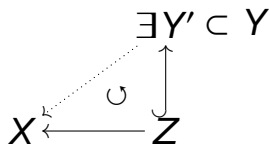
(Approx.) Absolute (Neighborhood) Retract

metric X or $(X, *)$: (A)A(N)R \iff
for all $Z \subset Y$ and for all $Z \rightarrow X$

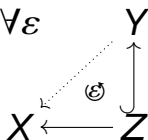
AR:



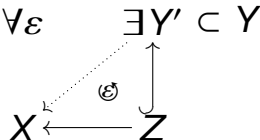
ANR:



AAR: $\forall \varepsilon$



AANR: $\forall \varepsilon$



(Approx.) Absolute (Neighborhood) Retract

$\mathcal{E}: \mathcal{C}pt \ni X \mapsto (X \amalg \{*\}, *) \in \mathcal{C}pt_*$

$X: (A)ANR \iff (X \amalg \{*\}, *): (A)ANR$

$(X \amalg \{*\}, *): \text{never AAR (hence never AR)}$

$\mathcal{F}: \mathcal{C}pt_* \ni (X, *) \mapsto X \in \mathcal{C}pt$

$(X, *): A(N)R \iff X: A(N)R$

$(X, *): AA(N)R \Rightarrow X: AA(N)R$

$\exists X \text{ AAR and } \exists *_1, *_2 \in X \text{ s.t.}$

$(X, *_1) \text{ AAR, } (X, *_2) \text{ not AANR}$

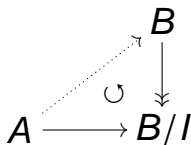
$(X, *): ANR \iff X: ANR$

$\iff (X \amalg \{*\prime\}, *'): ANR$

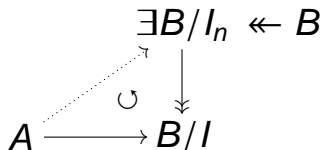
(Weakly) (Semi-)Projective C*-algebra

separable (unital) C*-algebra A : (W)(S)P \iff
 for all $B \rightarrow B/I$ and for all $A \rightarrow B/I$

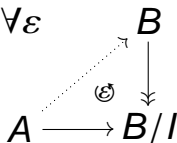
P:



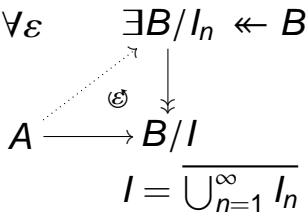
SP:



WP: $\forall \varepsilon$



WSP: $\forall \varepsilon$



Trivial shape, (A)AR and (W)P

X : contractible $\Rightarrow X$: *trivial shape*

Proposition

$X: AR \iff X: ANR$ and *trivial shape*

$\iff X: ANR$ and *contractible*

$X: AAR \iff X: AANR$ and *trivial shape*

A : contractible $\Rightarrow A$: *trivial shape*

Theorem (T.A. Loring, H. Thiel)

$A: P \iff A: SP$ and *trivial shape*

$\iff A: SP$ and *contractible*

$A: WP \iff A: WSP$ and *trivial shape*

Approximation problem in NC Shape theory

Theorem (T. A. Loring and T. Shulman)

The cone $C_0((0, 1], B)$ of a separable C^ -algebra B is an inductive limit of projective C^* -algebras.*

Theorem (H. Thiel)

A: separable C^ -algebra T.F.A.E*

- *A is an inductive limit of projective C^* -algs*
- *A is an inductive limit of cones*
- *A is an inductive limit of contractible C^* -algs*
- *A has trivial shape*

(cf. X contractible $\Rightarrow X$ inverse limit of ARs)

Commutative spaces in NC Shape theory

Lemma

$C_0(X, *) : (W)(S)P \Rightarrow (X, *) : (A)A(N)R$

$C(X) : (W)(S)P \Rightarrow X : (A)A(N)R$

D^2 : AR but $C(D^2)$: not even WSP

$$\begin{array}{ccc} & \bigoplus_{n=1}^{\infty} \mathcal{T} & \\ & \nearrow \times & \downarrow \\ C_0(D^2, 0) & \xrightarrow{\cup} & \bigoplus_{n=1}^{\infty} C(S^1) \end{array}$$

\mathcal{T} : Toeplitz algebra (NC 2-Disc)

$\mathcal{T} \twoheadrightarrow C(S^1)$

Commutative spaces in NC Shape theory

Theorem (Chigogidze-Dranishnikov,
Sørensen-Thiel, Enders)

$C_0(X, *) : (W)(S)P \iff (X, *) : (A)A(N)R$
and $\dim(X) \leq 1$

$C(X) : (W)(S)P \iff X : (A)A(N)R$ *and* $\dim(X) \leq 1$

Problem

Is every commutative C^ -algebra
a direct limit of SP C^* -algebras?*

Is there an obstruction in K_0 ?

Corollaries on Semiprojective (SP) C^* -algebras

Corollary

Y : locally compact

$F \subset Y$: finite set

$C_0(Y)$: SP \iff $C_0(Y \setminus F)$: SP

Corollary

Y : locally compact

n : integer

$M_n(C_0(Y))$: SP \iff $C_0(Y)$: SP

Extension problem

Question 1

When an extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0$$

is given with $\dim(F) < \infty$, then

I is semiprojective $\stackrel{?}{\iff}$ A is semiprojective

Yes if A is commutative (Sørensen-Thiel)

\Leftarrow is true (Enders)

\Rightarrow is not true in general (Eilers-Katsura)

Question 2

When B is a full corner in A (e.g. $A = M_2(B)$),
 A is semiprojective $\stackrel{?}{\Rightarrow} B$ is semiprojective

Yes if B is commutative and A is special form
(Sørensen-Thiel)

No in general (Eilers-Katsura)

Partial answers to 2 Questions are useful for

Problem

Is every C^ -alg a direct limit of SP C^* -algs?*