Shape theories of commutative and non-commutative spaces

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§0: Introduction

§1: (Approximative) Absolute (Neighborhood) Retract and (Weakly) (Semi-)Projective (abbreviated by (A)A(N)R and (W)(S)P)

§2: Commutative spaces in Noncommutative Shape theory

§3: Results on Noncommutative Shape theory
Homotopy theory is not perfect for all compact metric space $X$ approximate a space $X$ by nicer spaces ($=$ absolute neighborhood retracts (ANRs) $X_k$)

**Theorem**

*Every $X$ is an inverse limit of ANRs $(X_k)_k$:

$$\lim_{\leftarrow}(X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_k \leftarrow \cdots) \cong X$$

study approximating system $(X_k)_k$ instead of original space $X$
(e.g. shape equivalence)
Noncommutative shape theory

Noncommutative space = C*-algebra
(contravariantly)

approximate an arbitrary C*-algebra $A$ by
nicer (= semiprojective (SP)) C*-algebras $A_k$

Problem (Blackadar ’85)

Is every $A$ a direct limit of SP C*-algebras $(A_k)_k$?

$\lim_{\longrightarrow}(A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k \rightarrow \cdots ) \cong A$?

To solve this problem, we want lots of
examples of semiprojective C*-algebras.
Recent progress (after ’10)

- T. A. Loring and T. Shulman (§1)
- “Semiprojectivity and Asymptotic Morphisms” by S. Eilers and T. Shulman
- D. Enders (§3)
- A. P. W. Sørensen and H. Thiel (and D. Enders) (§2)
- S. Eilers and T. Katsura (§3)
- H. Thiel (§1)
- …
- …
Commutative spaces

Definition

\(\text{Cpt}\): the category of compact spaces
\(\text{Cpt}_\ast\): the category of pointed compact spaces

Definition

\(\mathcal{E}: \text{Cpt} \to \text{Cpt}_\ast\): Adding disjoint base points
\(\mathcal{F}: \text{Cpt}_\ast \to \text{Cpt}\): forget base points

Identify a pointed compact space \((X, \ast)\) with a locally compact space \(Y = X \setminus \{\ast\}\)

\((\text{Cpt}_\ast \ni (X, \ast) \mapsto X \setminus \{\ast\} \in \text{LocCpt} \) is “bijective”\)

\(\mathcal{E}: X \mapsto Y = X: \) compact space is locally compact
\(\mathcal{F}: Y \mapsto X = Y \cup \{\ast\}: \) one-point compactification
Commutative algebra $C_0(X, \ast)$

Definition

For $(X, \ast) \in \text{Cpt}_*$

$$C_0(X, \ast) := \{ f : X \rightarrow \mathbb{C} \mid \text{continuous, } f(\ast) = 0 \}$$

$C_0(X, \ast) = C_0(Y)$ for $Y = X \setminus \{ \ast \}$

$C_0(X, \ast)$: commutative $\mathbb{C}$-algebra

with involution $\ast$ and norm $\| \cdot \|_\infty$

$f^*(x) := f(x)$ for $f \in C_0(X, \ast), x \in X$

$\|f\|_\infty := \sup_{x \in X} |f(x)|$ for $f \in C_0(X, \ast)$
C*-algebras

Definition

C*-algebra = C-algebra with involution *
and norm \(|\cdot|\) satisfying ······

Definition

\(A, B: C^*\)-algebra
\(\varphi: A \rightarrow B: \ast\)-homomorphism \iff
\(\varphi: C\)-algebra hom preserving involution *

Definition

\(C^{*}\text{-alg}\): Category of C*-algebras
\(C^{*}\text{-alg}_1\): Category of unital C*-algebras
Theorem

\[ \text{Cpt}_* \ni (X, *) \mapsto C_0(X, *) \in C^*\text{-alg} \]

(and \( \text{Cpt} \ni X \mapsto C(X) \in C^*\text{-alg}_1 \)): “isomorphism” onto commutative (unital) \( C^*\)-algebras

\( C^*\text{-alg}^{op} \) (resp. \( C^*\text{-alg}_1^{op} \)) can be called category of NC locally compact (resp. compact) spaces

\[ \mathcal{E} : C^*\text{-alg}_1 \ni A \mapsto A \in C^*\text{-alg} : \text{forget the unit} \]

\[ \mathcal{F} : C^*\text{-alg} \ni A \mapsto A^+ \in C^*\text{-alg}_1 : \text{add the unit (= NC one-point compactification)} \]
metric $X$ or $(X, \ast)$: (A)A(N)R $\iff$
for all $Z \subset Y$ and for all $Z \to X$

AR: $Y$

ANR: $\exists Y' \subset Y$

AAR: $\forall \varepsilon$

AANR: $\forall \varepsilon$ $\exists Y' \subset Y$
(Approx.) Absolute (Neighborhood) Retract

\[ \mathcal{E} : \text{Cpt} \ni X \mapsto (X \sqcup \{\ast\}, \ast) \in \text{Cpt}_* \]

\[ X : (\text{A})\text{ANR} \iff (X \sqcup \{\ast\}, \ast) : (\text{A})\text{ANR} \]

\((X \sqcup \{\ast\}, \ast) : \text{never } \text{AAR (hence never } \text{AR)}\]

\[ \mathcal{F} : \text{Cpt}_* \ni (X, \ast) \mapsto X \in \text{Cpt} \]

\[ (X, \ast) : \text{A(N)R} \iff X : \text{A(N)R} \]

\[ (X, \ast) : \text{AA(N)R} \Rightarrow X : \text{AA(N)R} \]

\[ \exists X \text{ AAR and } \exists \ast_1, \ast_2 \in X \text{ s.t.} \]

\[ (X, \ast_1) \text{ AAR, } (X, \ast_2) \text{ not AANR} \]

\[ (X, \ast) : \text{ANR} \iff X : \text{ANR} \iff (X \sqcup \{\ast'\}, \ast') : \text{ANR} \]
(Weakly) (Semi-)Projective C*-algebra

separable (unital) C*-algebra $A : (W)(S)P \iff$ for all $B \to B/I$ and for all $A \to B/I$

$P:$

$\begin{array}{c}
A \to B/I \\
\cup \\
A \to B/I
\end{array}$

$SP:$

$\begin{array}{c}
A \to B/I \\
\exists B/I_n \leftrightarrow B
\end{array}$

$WP:$

$\begin{array}{c}
A \to B/I \\
\varepsilon \\
A \to B/I
\end{array}$

$WSP:$

$\begin{array}{c}
A \to B/I \\
\varepsilon \\
\exists B/I_n \leftrightarrow B
\end{array}$

$I = \bigcup_{n=1}^{\infty} I_n$
Trivial shape, (A)AR and (W)P

\[ X: \text{contractible} \implies X: \text{trivial shape} \]

**Proposition**

\[ X: \text{AR} \iff X: \text{ANR and trivial shape} \]
\[ \iff X: \text{ANR and contractible} \]
\[ X: \text{AAR} \iff X: \text{AANR and trivial shape} \]

\[ A: \text{contractible} \implies A: \text{trivial shape} \]

**Theorem (T.A. Loring, H. Thiel)**

\[ A: \text{P} \iff A: \text{SP and trivial shape} \]
\[ \iff A: \text{SP and contractible} \]
\[ A: \text{WP} \iff A: \text{WSP and trivial shape} \]
Theorem (T. A. Loring and T. Shulman)

The cone $C_0((0,1], B)$ of a separable $C^*$-algebra $B$ is an inductive limit of projective $C^*$-algebras.

Theorem (H. Thiel)

$A$: separable $C^*$-algebra T.F.A.E

- $A$ is an inductive limit of projective $C^*$-algs
- $A$ is an inductive limit of cones
- $A$ is an inductive limit of contractible $C^*$-algs
- $A$ has trivial shape

(cf. $X$ contractible $\Rightarrow X$ inverse limit of ARs)
Lemma

\[ C_0(X, \ast): (W)(S)P \Rightarrow (X, \ast): (A)A(N)R \]
\[ C(X): (W)(S)P \Rightarrow X: (A)A(N)R \]

\( D^2: \text{AR} \quad \text{but} \quad C(D^2): \text{not even WSP} \)

\[ \bigoplus_{n=1}^{\infty} \mathcal{T} \]

\[ C_0(D^2, 0) \rightarrow \bigoplus_{n=1}^{\infty} C(S^1) \]

\( \mathcal{T}: \text{Toeplitz algebra (NC 2-Disc)} \quad \mathcal{T} \rightarrow C(S^1) \)
Commutative spaces in NC Shape theory

Theorem (Chigogidze-Dranishnikov, Sørensen-Thiel, Enders)

\[ C_0(X, \ast): (W)(S)P \iff (X, \ast): (A)A(N)R \]
\[ \text{and } \dim(X) \leq 1 \]
\[ C(X): (W)(S)P \iff X: (A)A(N)R \text{ and } \dim(X) \leq 1 \]

Problem

Is every commutative $C^*$-algebra a direct limit of SP $C^*$-algebras?

Is there an obstruction in $K_0$?
Corollaries on Semiprojective (SP) C*-algebras

Corollary

\[ Y: \text{locally compact} \]
\[ F \subset Y: \text{finite set} \]
\[ C_0(Y): \text{SP} \iff C_0(Y \setminus F): \text{SP} \]

Corollary

\[ Y: \text{locally compact} \]
\[ n: \text{integer} \]
\[ M_n(C_0(Y)): \text{SP} \iff C_0(Y): \text{SP} \]
Question 1

When an extension

\[
0 \rightarrow I \rightarrow A \rightarrow F \rightarrow 0
\]

is given with \( \dim(F) < \infty \), then

\[?
I \text{ is semiprojective} \iff A \text{ is semiprojective}\]

Yes if \( A \) is commutative (Sørensen-Thiel)
\(\iff\) is true (Enders)
\(\Rightarrow\) is not true in general (Eilers-Katsura)
**Question 2**

When $B$ is a full corner in $A$ (e.g. $A = M_2(B)$),

$A$ is semiprojective $\Rightarrow B$ is semiprojective

Yes if $B$ is commutative and $A$ is special form
(Sørensen-Thiel)

No in general (Eilers-Katsura)

Partial answers to 2 Questions are useful for

**Problem**

*Is every C*-alg a direct limit of SP C*-alg?*