Shape theories of commutative and non-commutative spaces

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Plan

§0: Introduction

§1: (Approximative) Absolute (Neighborhood) Retract and (Weakly) (Semi-)Projective (abbreviated by (A)A(N)R and (W)(S)P)

§2: Commutative spaces in Noncommutative Shape theory

§3: Results on Noncommutative Shape theory

Commutative shape theory

Homotopy theory is not perfect
for all compact metric space X

approximate a space X by nicer spaces
(= absolute neighborhood retracts (ANRs) X_k)

Theorem

Every X is an inverse limit of ANRs $(X_k)_k$:

$$\varprojlim(X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_k \leftarrow \cdots) \cong X$$

study approximating system $(X_k)_k$
instead of original space X
(e.g. shape equivalence)

Noncommutative shape theory

Noncommutative space = C*-algebra
(contravariantly)

approximate an arbitrary C*-algebra A by
nicer (= semiprojective (SP)) C*-algebras A_k

Problem (Blackadar '85)

Is every A a direct limit of SP C-algebras $(A_k)_k$?*
$$\varinjlim (A_1 \to A_2 \to \cdots \to A_k \to \cdots) \cong A?$$

To solve this problem, we want lots of
examples of semiprojective C*-algebras.

Recent progress (after ’10)

- T. A. Loring and T. Shulman (§1)
- “Semiprojectivity and Asymptotic Morphisms” by S. Eilers and T. Shulman
- D. Enders (§3)
- A. P. W. Sørensen and H. Thiel (and D. Enders) (§2)
- S. Eilers and T. Katsura (§3)
- H. Thiel (§1)
- $\cdots$
- T. A. Loring, U. Otogonbayar, T. Shulman, A. P. W. Sørensen, T. Katsura and $\cdots$
- $\cdots$

Commutative spaces

Definition

$\mathfrak{Cpt}$: the category of compact spaces
$\mathfrak{Cpt}_*$: the category of pointed compact spaces

Definition

$\mathcal{E}$: $\mathfrak{Cpt} \to \mathfrak{Cpt}_*$: Adding disjoint base points
$\mathcal{F}: \mathfrak{Cpt}_* \to \mathfrak{Cpt}$: forget base points

Identify a pointed compact space $(X, *)$
with a locally compact space $Y = X \setminus \{*\}$
($\mathfrak{Cpt}_* \ni (X, *) \mapsto X \setminus \{*\} \in \mathfrak{LocCpt}$ is "bijective")
$\mathcal{E}: X \mapsto Y = X$: compact space is locally compact
$\mathcal{F}: Y \mapsto X = Y \cup \{*\}$: one-point compactification

Commutative algebra $C_0(X,*)$

Definition

For $(X,*) \in \mathfrak{Cpt}_*$
$$C_0(X,*) := \{f : X \to \mathbb{C} \mid \text{continuous}, f(*) = 0\}$$

$C_0(X,*) = C_0(Y)$ for $Y = X \setminus \{*\}$

$C_0(X,*)$: commutative $\mathbb{C}$-algebra with involution * and norm $\|\cdot\|_\infty$

$f^*(x) := \overline{f(x)}$ for $f \in C_0(X,*)$, $x \in X$

$\|f\|_\infty := \sup_{x\in X} |f(x)|$ for $f \in C_0(X,*)$

C*-algebras

Definition

C*-algebra = $\mathbb{C}$-algebra with involution *
and norm $\|\cdot\|$ satisfying $\cdots\cdots$

Definition

A, B: C*-algebra
$\varphi\colon A \to B$: *-homomorphism $\iff$
φ: $\mathbb{C}$-algebra hom preserving involution *

Definition

$\mathfrak{C}$*-$\mathfrak{alg}$: Category of C*-algebras
$\mathfrak{C}$*-$\mathfrak{alg}_1$: Category of unital C*-algebras

Noncommutative (NC) spaces

Theorem

$\mathfrak{Cpt}_* \ni (X,*) \mapsto C_0(X,*) \in \mathfrak{C}^*\text{-}\mathfrak{alg}$ (and $\mathfrak{Cpt} \ni X \mapsto C(X) \in \mathfrak{C}^*\text{-}\mathfrak{alg}_1$): "isomorphism" onto commutative (unital) C-algebras*

$\mathfrak{C}^*\text{-}\mathfrak{alg}^{op}$ (resp. $\mathfrak{C}^*\text{-}\mathfrak{alg}_1^{op}$) can be called category of NC locally compact (resp. compact) spaces

$\mathcal{E} : \mathfrak{C}^*\text{-}\mathfrak{alg}_1 \ni A \mapsto A \in \mathfrak{C}^*\text{-}\mathfrak{alg}$: forget the unit

$\mathcal{F} : \mathfrak{C}^*\text{-}\mathfrak{alg} \ni A \mapsto A^+ \in \mathfrak{C}^*\text{-}\mathfrak{alg}_1$:
add the unit (= NC one-point compactification)

(Approx.) Absolute (Neighborhood) Retract

metric X or $(X, *)$: (A)A(N)R $\iff$
for all $Z \subset Y$ and for all $Z \to X$

AR:
$$\begin{tikzcd} & Y \arrow[dl, dotted] \\ X & Z \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

ANR:
$$\begin{tikzcd} & \exists Y' \subset Y \arrow[dl, dotted] \\ X & Z \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

AAR: $\forall \varepsilon$
$$\begin{tikzcd} & Y \arrow[dl, dotted] \\ X & Z \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft_{\varepsilon}"] \end{tikzcd}$$

AANR: $\forall \varepsilon$
$$\begin{tikzcd} & \exists Y' \subset Y \arrow[dl, dotted] \\ X & Z \arrow[l] \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft_{\varepsilon}"] \end{tikzcd}$$

(Approx.) Absolute (Neighborhood) Retract

$\mathcal{E}\colon \mathfrak{Cpt} \ni X \mapsto (X \amalg \{*\}, *) \in \mathfrak{Cpt}_*$

X: (A)ANR $\iff$ $(X \amalg \{*\}, *)$: (A)ANR

$(X \amalg \{*\}, *)$: never AAR (hence never AR)

$\mathcal{F}\colon \mathfrak{Cpt}_* \ni (X, *) \mapsto X \in \mathfrak{Cpt}$

$(X, *)$: A(N)R $\iff$ X: A(N)R

$(X, *)$: AA(N)R $\Rightarrow$ X: AA(N)R

$\exists X$ AAR and $\exists *_1, *_2 \in X$ s.t.
$(X, *_1)$ AAR, $(X, *_2)$ not AANR

$(X, *)$: ANR $\iff$ X: ANR
$\iff$ $(X \amalg \{*'\}, *')$: ANR

(Weakly) (Semi-)Projective C*-algebra

separable (unital) C*-algebra A : (W)(S)P $\iff$
for all $B \to B/I$ and for all $A \to B/I$

P:

$$\begin{tikzcd}
& B \arrow[d, two heads] \\
A \arrow[ur, dotted] \arrow[r] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] & B/I
\end{tikzcd}$$

SP:

$$\begin{tikzcd}
& \exists B/I_n \arrow[d, two heads] & B \arrow[l, two heads] \\
A \arrow[ur, dotted] \arrow[r] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] & B/I &
\end{tikzcd}$$

WP: $\forall \varepsilon$

$$\begin{tikzcd}
& B \arrow[d, two heads] \\
A \arrow[ur, dotted] \arrow[r] \arrow[ur, phantom, shift left=3, "\varepsilon"] & B/I
\end{tikzcd}$$

WSP: $\forall \varepsilon$

$$\begin{tikzcd}
& \exists B/I_n \arrow[d, two heads] & B \arrow[l, two heads] \\
A \arrow[ur, dotted] \arrow[r] \arrow[ur, phantom, shift left=3, "\varepsilon"] & B/I &
\end{tikzcd}$$

$$I = \overline{\bigcup_{n=1}^{\infty} I_n}$$

Trivial shape, (A)AR and (W)P

X: contractible $\Rightarrow$ X: *trivial shape*

Proposition

X: AR $\iff$ X: ANR and trivial shape
$\iff$ X: ANR and contractible
X: AAR $\iff$ X: AANR and trivial shape

A: contractible $\Rightarrow$ A: *trivial shape*

Theorem (T.A. Loring, H. Thiel)

A: P $\iff$ A: SP and trivial shape
$\iff$ A: SP and contractible
A: WP $\iff$ A: WSP and trivial shape

Approximation problem in NC Shape theory

Theorem (T. A. Loring and T. Shulman)

The cone $C_0((0,1], B)$ of a separable C-algebra B is an inductive limit of projective C*-algebras.*

Theorem (H. Thiel)

A: separable C-algebra T.F.A.E*

- *A is an inductive limit of projective C*-algs*
- *A is an inductive limit of cones*
- *A is an inductive limit of contractible C*-algs*
- *A has trivial shape*

(cf. X contractible $\Rightarrow$ X inverse limit of ARs)

Commutative spaces in NC Shape theory

Lemma

$C_0(X,*)$: *(W)(S)P* $\Rightarrow$ $(X,*)$: *(A)A(N)R*
$C(X)$: *(W)(S)P* $\Rightarrow$ X: *(A)A(N)R*

D^2: AR but $C(D^2)$: not even WSP

$$\begin{tikzcd} & \bigoplus_{n=1}^{\infty} \mathcal{T} \arrow[d, two heads] \\ C_0(D^2,0) \arrow[r] \arrow[ur, dotted, "\times" description] \arrow[ur, phantom, shift left=4ex, "\circlearrowleft"] & \bigoplus_{n=1}^{\infty} C(S^1) \end{tikzcd}$$

$\mathcal{T}$: Toeplitz algebra (NC 2-Disc) $\mathcal{T} \twoheadrightarrow C(S^1)$

Commutative spaces in NC Shape theory

Theorem (Chigogidze-Dranishnikov, Sørensen-Thiel, Enders)

$C_0(X,*)$: *(W)(S)P* $\iff$ $(X,*)$: *(A)A(N)R* *and* $\dim(X) \leq 1$

$C(X)$: *(W)(S)P* $\iff$ X: *(A)A(N)R and* $\dim(X) \leq 1$

Problem

Is every commutative C-algebra a direct limit of SP C*-algebras?*

Is there an obstraction in K_0?

Corollaries on Semiprojective (SP) C*-algebras

Corollary

Y: locally compact

$F \subset Y$: finite set

$C_0(Y)$: *SP* $\iff$ $C_0(Y \setminus F)$: *SP*

Corollary

Y: locally compact

n: integer

$M_n(C_0(Y))$: *SP* $\iff$ $C_0(Y)$: *SP*

Extension problem

Question 1

When an extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0$$

is given with $\dim(F) < \infty$, then

$$I \text{ is semiprojective } \overset{?}{\Longleftrightarrow} A \text{ is semiprojective}$$

Yes if A is commutative (Sørensen-Thiel)
$\Leftarrow$ is true (Enders)
$\Rightarrow$ is not true in general (Eilers-Katsura)

full corner

Question 2

When B is a full corner in A (e.g. $A = M_2(B)$),
A is semiprojective $\overset{?}{\Rightarrow}$ B is semiprojective

Yes if B is commutative and A is special form (Sørensen-Thiel)
No in general (Eilers-Katsura)

Partial answers to 2 Questions are useful for

Problem

Is every C-alg a direct limit of SP C*-algs?*