Euler characteristics of small categories

With appiclations to p-subgroup categories

Martin Wedel Jacobsen and Jesper Michael Møller

University of Copenhagen

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Outline of talk

1. Euler characteristics of square matrices
   - Euler characteristic of the poset $\mathcal{S}_G^*$
   - Euler characteristic of the fusion category $\mathcal{F}_G^*$
   - Euler characteristic of the orbit category $\mathcal{O}_G^*$
2. Möbius algebras

The Euler characteristic of a category $\mathcal{C}$ only depends on $\zeta(\mathcal{C})$

$$\mathcal{C} \to \zeta(\mathcal{C}) \to \begin{matrix} k_{\mathcal{C}}^{\bullet} : \mathcal{C} \to \mathbf{Q} \\ k^{\mathcal{C}}_{\bullet} : \mathcal{C} \to \mathbf{Q} \end{matrix} \to \sum_b k_{\mathcal{C}}^b = \chi(\mathcal{C}) = \sum_a k_a^{\mathcal{C}}$$

Example (The Euler characteristic of a one-object category)

$$\chi(G) = |G|^{-1}$$

Summary of main results [2]

$\mathcal{S}_G$ Brown poset of p-subgroups
$\mathcal{F}_G$ Fusion category of p-subgroups
$\mathcal{O}_G$ Orbit category of p-subgroups

$\mathcal{C}$	$\chi(\mathcal{C})$	$\chi(\mathcal{C})$
$\mathcal{S}_G^*$	$\sum_{[H]>1} -\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})$	$\sum_{K>1} -\mu(K)$
$\mathcal{F}_G^*$	$\sum_{[H]>1} \sum_{x \in C_{N_G(H)}(H)} \frac{-\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H})}{\lvert N_G(H)\rvert}$	$\sum_{[K]>1} \frac{-\mu(K)}{\lvert\mathcal{F}_G^*(K)\rvert}$
$\mathcal{O}_G$	$\sum_{[H]\geq 1} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{\lvert\mathcal{O}_G(H)\rvert}$	$\frac{1+(p-1)\sum \lvert C\rvert}{p\lvert G\rvert}$

Weightings and coweightings for a square matrix ζ

Definition

A weighting for ζ is a column vector $(k_\zeta^\bullet)$ such that

$$(\zeta(a,b))(k_\zeta^b) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

A coweighting for ζ is a row vector $(k_\bullet^\zeta)$ such that

$$(k_a^\zeta)(\zeta(a,b)) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$$

- A matrix may have none or many (co)weightings
- If $(\mu(a,b))$ is an inverse to $(\zeta(a,b))$ then

$$(k^a_\zeta) = (\mu(a,b)) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \big(\sum_b \mu(a,b)\big) \quad \text{(row sums)}$$

$$(k^\zeta_b) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} (\mu(a,b)) = \big(\sum_a \mu(a,b)\big) \quad \text{(column sums)}$$

  are the unique weighting and coweighting for ζ
- $\zeta = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (not invertible) has weighting and coweighting
- $\zeta = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ has a weighting but no coweighting

If ζ admits both a weighting $k_\zeta^\bullet$ and a coweighting $k_\bullet^\zeta$ then the sum of their values agree

$$\sum_b k_\zeta^b = \sum_b \Big(\sum_a k_a^\zeta \zeta(a,b) \Big) k_\zeta^b = \sum_a k_a^\zeta \Big(\sum_b \zeta(a,b) k_\zeta^b \Big) = \sum_a k_a^\zeta$$

Definition (The Euler characteristic of a matrix (Leinster 2008))

$$\sum_b k_\zeta^b = \chi(\zeta) = \sum_a k_a^\zeta$$

If ζ is invertible then

$$\chi(\zeta) = \sum_b k_\zeta^b = \sum_{a,b} \mu(\mathcal{C})(a,b) = \sum_a k_a^\zeta, \quad \mu(\mathcal{C}) = \zeta(\mathcal{C})^{-1}$$

Definition (The incidence matrix of a finite category $\mathcal{C}$)

$$\zeta(\mathcal{C}) = \big(\zeta(a,b)\big)_{a,b\in\mathcal{C}} \qquad \zeta(a,b) = |\mathcal{C}(a,b)|$$

Definition ((Reduced) Euler characteristic of a category via $\zeta(\mathcal{C})$)

$$\chi(\mathcal{C}) = \chi(\zeta(\mathcal{C})) = \sum_b k^b_{\zeta(\mathcal{C})} = \sum_a k_a^{\zeta(\mathcal{C})}$$

$$\widetilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1$$

Proposition (Invarians under equivalence (Leinster 2008 [3]))

- *If there is an adjunction* $\mathcal{C} \rightleftarrows \mathcal{D}$ *then* $\chi(\mathcal{C}) = \chi(\mathcal{D})$
- *If* $\mathcal{C}$ *has an initial or terminal element then* $\chi(\mathcal{C}) = 1$
- *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are equivalent then* $\chi(\mathcal{C}) = \chi(\mathcal{D})$

Euler characteristic of a finite poset

Definition (Incidence and Möbius matrix of a finite poset $\mathcal{S}$)

$$\zeta(\mathcal{S}) = \big(\zeta(a,b)\big)_{a,b\in\mathcal{S}} \qquad \zeta(a,b) = \begin{cases} 1 & a \leq b \\ 0 & \text{otherwise} \end{cases}$$

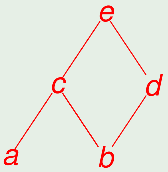
$$\mu(\mathcal{S}) = \zeta(\mathcal{S})^{-1}$$

Definition (The Euler characteristic of a finite poset $\mathcal{S}$)

$$\chi(\mathcal{S}) = \chi(\zeta(\mathcal{S})) = \sum_{a,b\in\mathcal{S}} \mu(\mathcal{S})(a,b) = \sum_b k_{\mathcal{S}}^b = \sum_a k_a^{\mathcal{S}}$$

$$k_{\mathcal{S}}^b = \sum_a \mu(\mathcal{S})(a,b) \qquad k_a^{\mathcal{S}} = \sum_b \mu(\mathcal{S})(a,b)$$

Example (A poset with a terminal element)



$$\zeta(\mathcal{S}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mu(\mathcal{S}) = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\chi(\mathcal{S}) = \chi(\zeta(\mathcal{S})) = 1$$

Question

What is the relation between the combinatorial Euler characteristic $\chi(\mathcal{S})$ and the topological Euler characteristic $\chi(B\mathcal{S})$?

Definition (Simplices in a poset)

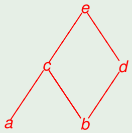
A k-simplex, $k \geq 0$, (from a to b) is a totally ordered subset of $k+1$ points (with a as smallest and b as greatest element).

Example ($(\zeta - E)^k$ counts k-simplices)

0-simplices $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\zeta - E)^0$$

1-simplices $$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\zeta - E)^1$$

2-simplices $$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (\zeta - E)^2$$



Lemma (Counting simplices in poset $\mathcal{S}$)

$(\zeta - E)^k(a,b) = \#\{k\text{-simplices from } a \text{ to } b\}$ $(k \geq 0)$
$\sum_{a,b}(\zeta - E)^k(a,b) = \#\{k\text{-simplices in } \mathcal{S}\}$ $(k \geq 0)$

Topological Euler characteristic of the realization $B\mathcal{S}$

$$\begin{aligned}\chi(B\mathcal{S}) &= \sum_{k=0}^{\infty}(-1)^k \#\{k\text{-simplices in } \mathcal{S}\} \\ &= \sum_{k=0}^{\infty}(-1)^k \sum_{a,b\in\mathcal{S}}(\zeta - E)^k(a,b) \\ &= \sum_{a,b}\sum_{k=0}^{\infty}(-1)^k(\zeta - E)^k(a,b) \\ &= \sum_{a,b}\zeta^{-1}(a,b) = \sum_{a,b}\mu(a,b) = \chi(\mathcal{S})\end{aligned}$$

$x^{-1} = (1 + (x-1))^{-1} = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$

Lemma ($\mu(a, b)$ depends only on the interval $[a, b]$)

$$\mu(a,b) = \begin{cases} 1 & a = b \\ \widetilde{\chi}(a,b) & a < b \\ 0 & a \not\leq b \end{cases}$$

Proof in case $a < b$

$$\begin{aligned} \mu(a,b) &= \sum_{k=0}^{\infty} (-1)^k (\zeta - E)^k (a,b) \\ &= 0 + (-1) + \sum_{k=2}^{\infty} (-1)^k (\zeta - E)^k (a,b) \\ &= -1 + \sum_{k=0}^{\infty} \#\{k\text{-simplices in } (a,b)\} = \widetilde{\chi}(a,b) \end{aligned}$$

The Brown subgroup posets $\mathcal{S}_G$ and $\mathcal{S}_G^*$ at p

Assumptions

G is a finite group and p is a prime number

Definition (The posets $\mathcal{S}_G$ and $\mathcal{S}_G^*$)

- $\mathcal{S}_G$ is the poset of all p-subgroups of G
- $\mathcal{S}_G^*$ is the poset of nonidentity p-subgroups of G

- $\chi(\mathcal{S}_G) = 1$ because $\mathcal{S}_G$ has initial element 1
- What is $\chi(\mathcal{S}_G^*)$? Find a weighting and a coweighting!

Definition ($\mu(K)$ for finite p-group K)

$$\mu(K) = \begin{cases} (-1)^n p^{\binom{n}{2}} & K \text{ elementary abelian, } |K| = p^n \\ 0 & \text{otherwise} \end{cases}$$

(Co)Weighting and Euler characteristic for $\mathcal{S}_G$ and $\mathcal{S}_G^*$

$$k_{\mathcal{S}}^H = -\widetilde{\chi}(\mathcal{S}^*_{N_G(H)/H}), \qquad k_K^{\mathcal{S}} = -\mu(K)$$

$$\sum_{H>1} -\widetilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = \chi(\mathcal{S}_G^*) = \sum_{K>1} -\mu(K)$$

Reformulation

$$\sum_{H\geq 1} -\widetilde{\chi}(\mathcal{S}^*_{N_G(H)/H}) = 1, \qquad \sum_{K\geq 1} \mu(K) = -\widetilde{\chi}(\mathcal{S}_G^*)$$

The poset $\mathcal{S}_G^*$ knows

- the elementary abelian p-subgroups
- the p-radical p-subgroups (Strong Quillen Conjecture!)

Example (Symmetric and alternating groups at $p = 2$)

n	4	5	6	7	8	9	10
$\chi(\mathcal{S}_{S_n}^*)$	1	-15	-15	161	513	-639	-7935
$\chi(\mathcal{S}_{A_n}^*)$	1	5	-15	-175	65	5121	15105

Example (Alternating groups at $p = 3$)

n	4	5	6	7	8	9	10
$\chi(\mathcal{S}_{A_n}^*)$	4	10	10	-35	-224	-2996	-24380

What is known about $\chi(\mathcal{S}_G^*)$?

Theorem (Product formula)

$$-\widetilde{\chi}(\mathcal{S}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{S}^*_{G_i})$$

Theorem (Brown 1975, Quillen 1978)

$-\widetilde{\chi}(\mathcal{S}_G^*)$ is divisible by $|G|_p$

Theorem (Quillen 1978)

If G has a nonidentity normal p-subgroup then $\mathcal{S}_G^* \simeq *$

Strong Quillen Conjecture

$$O_pG > 1 \iff \widetilde{\chi}(\mathcal{S}_G^*) = 0$$

What is unknown about $\chi(\mathcal{S}_G^*)$!

- (Euler characteristics of Chevalley groups) Is $-\widetilde{\chi}(\mathcal{S}_{G_n(q)}^*) = (-1)^n q^{\#\{\text{positive roots}\}}$ for $G = A, B, C, D, E$?
- (Euler characteristics of alternating groups) Describe the sequence $n \to \chi(\mathcal{S}_{A_n}^*)$
- (Original Quillen conjecture 1978) $O_pG > 1 \iff \mathcal{S}_G^* \simeq *$
- (Strong Quillen conjecture) $O_pG > 1 \iff \widetilde{\chi}(\mathcal{S}_G^*) = 0$

The p-subgroup categories $\mathcal{F}_G$ and $\mathcal{F}_G^*$

Assumptions

G is a finite group and p is a prime number

The fusion categories $\mathcal{F}_G$ and $\mathcal{F}_G^*$ at p

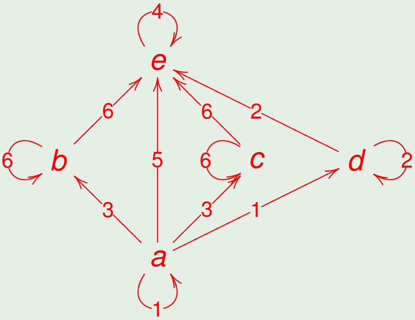
- $\mathcal{F}_G$ is the fusion category of all p-subgroups of G
- $\mathcal{F}_G^*$ is the fusion category of nonidentity p-subgroups of G

- $\mathcal{F}_G$ is a finite category with morphism sets

$$\mathcal{F}_G(H,K) = C_G(H)\backslash N_G(H,K), \quad \mathcal{F}_G(H) = C_G(H)\backslash N_G(H)$$

- $\chi(\mathcal{F}_G) = 1$ as $\mathcal{F}_G$ has initial element 1
- What is $\chi(\mathcal{F}_G^*)$? We need a weighting and a coweighting!

Example (Skeletal subcategory of $\mathcal{F}_{A_6}^*$, $p = 2$)



$$\zeta(\mathcal{F}_{A_6}^*) = \begin{pmatrix} 1 & 3 & 3 & 1 & 5 \\ 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$k^\bullet_{\mathcal{F}_{A_6}^*} = \begin{pmatrix} 0 \\ -1/12 \\ -1/12 \\ 1/4 \\ 1/4 \end{pmatrix} \qquad k_\bullet^{\mathcal{F}_{A_6}^*} = \begin{pmatrix} 1 & -1/3 & -1/3 & 0 & 0 \end{pmatrix} \qquad \chi(\mathcal{F}_{A_6}^*) = 1/3$$

(Co)Weighting and Euler characteristic of fusion category $\mathcal{F}_G^*$

$$k_{\mathcal{F}}^H = \frac{1}{|G|} \sum_{x \in C_G(H)} -\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H}), \quad k_K^{\mathcal{F}} = \frac{1}{|G|}\mu(K)|C_G(K)|$$

$$\frac{1}{|G|} \sum_{H>1} \sum_{x \in C_G(H)} -\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H}) = \chi(\mathcal{F}_G^*) = \sum_{[K]>1} \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|}$$

Reformulation

$$\sum_{H \geq 1} \sum_{x \in C_G(H)} -\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H}) = |G|, \quad \sum_{[K] \geq 1} \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|} = \widetilde{\chi}(\mathcal{F}_G^*)$$

- The category $\mathcal{F}_G^*$ knows the elementary abelian p-subgroups

What is known about $\chi(\mathcal{F}_G^*)$?

Theorem (Product formula)

$$-\widetilde{\chi}(\mathcal{F}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{F}^*_{G_i})$$

Proposition

- *If G has a nonidentity central p-subgroup then* $\widetilde{\chi}(\mathcal{F}_G^*) = 0$
- $|G|_{p'} \cdot \chi(\mathcal{F}_G^*) \in \mathbf{Z}$
- $\chi(\mathcal{F}_G^*) = \dfrac{|\{\varphi \in \mathcal{F}_G^*(P) \mid C_P(\varphi) > 1\}|}{|\mathcal{F}_G^*(P)|}$ *when P, the Sylow p-subgroup, is abelian.*
- $\chi(\mathcal{F}_G^*) = \chi(\mathcal{F}_G^{\mathrm{a}})$ *and* $\chi(\mathcal{F}_G^*) = \chi(\widetilde{\mathcal{F}}_G^*)$

Example (Alternating groups A_n at $p = 2$)

n	$\chi(\mathcal{S}_{A_n}^*)$	$\chi(\mathcal{F}_{A_n}^*)$
4	1	$1/3$
5	5	$1/3$
6	-15	$1/3$
7	-175	$1/3$
8	68	$41/63$
9	5121	$41/63$

n	$\chi(\mathcal{S}_{A_n}^*)$	$\chi(\mathcal{F}_{A_n}^*)$
10	55105	$18/35$
11	55935	$18/35$
12	-288255	$389/567$
13	1626625	$389/567$
14	23664641	$233/405$
15	150554625	$233/405$

Example (The smallest group with $\chi(\mathcal{F}_G^*) > 1$)

There is a group $G = C_2^4 \rtimes H$, where $H = (C_3 \times C_3) \rtimes C_2$, of order $|G| = 288$ with Euler characteristic $\chi(\mathcal{F}_G^*) = 10/9$ at $p = 2$.

What is unknown about $\chi(\mathcal{F}_G^*)$

- Are $\mathcal{F}_G^*$ and $\mathcal{F}_G^a$ homotopy equivalent?
- Are $\mathcal{F}_G^*$ and $\widetilde{\mathcal{F}}_G^*$ homotopy equivalent?
- Is $\chi(\mathcal{F}_G^*)$ always positive when p divides the order of G?
- Can $\chi(\mathcal{F}_G^*)$ get arbitrarily large?
- What is $\chi(\mathcal{F}_{A_n}^*)$? Does it converge for $n \to \infty$?
- What is $\chi(\mathcal{F}_{\mathrm{SL}_n(\mathbf{F}_q)}^*)$?
- Is there a $|G|_{p'}$-fold covering map $E \to B\mathcal{F}_G^*$ where E is (homotopy) finite and $\chi(E) = |G|_{p'}\chi(\mathcal{F}_G^*)$?

Categories of centric subgroups

Definition

The p-subgroup $H \leq G$ is p-centric if $p \nmid |C_G(H) \colon C_H(H)|$

Example (Euler characteristics of centric subgroup categories for alternating groups at $p = 2$)

n	4	5	6	7	8	9	10	11
$\lvert A_n\rvert \chi(\mathcal{L}^c_{A_n})$	1	5	-15	-105	65	585	11745	129195
$\chi(\mathcal{S}^c_{A_n})$	1	5	-15	-175	65	585	11745	107745
$\chi(\mathcal{L}^c_{A_n})$	$1/12$		$-1/24$		$13/4032$		$29/4480$	
$\chi(\mathcal{F}^c_{A_n})$	$1/3$		$1/3$		$13/63$		$19/105$	
$\chi(\widetilde{\mathcal{F}}^c_{A_n})$	$1/3$		$1/3$		$13/63$		$19/105$	

Weighting and Euler characteristic for $\widetilde{\mathcal{F}}_G^c$

$$|G\colon N_G(H)|k^H_{\widetilde{\mathcal{F}}_G^c} = \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^c(H)})}{|\widetilde{\mathcal{F}}_G^c(H)|} \in \mathbf{Z}_{(p)}$$

$$\chi(\widetilde{\mathcal{F}}_G^c) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^c(H)})}{|\widetilde{\mathcal{F}}_G^c(H)|}$$

- The category $\widetilde{\mathcal{F}}_G^c$ knows the p-selfcentralizing $\mathcal{F}_G$-radical p-subgroups

Conjecture

$$\chi(\mathcal{F}_G^c) = \chi(\widetilde{\mathcal{F}}_G^c)$$

The p-subgroup categories $\mathcal{O}_G$ and $\mathcal{O}_G^*$

Assumptions

G is a finite group and p is a prime number

The orbit categories $\mathcal{O}_G$ and $\mathcal{O}_G^*$ at p

- $\mathcal{O}_G$ is the orbit category of all p-subgroups of G
- $\mathcal{O}_G^*$ is the orbit category of nonidentity p-subgroups of G

- $\mathcal{O}_G$ is a finite category with morphism sets

$$\mathcal{O}_G(H,K) = N_G(H,K)/K, \quad \mathcal{O}_G(H) = N_G(H)/H$$

Weighting and Euler characteristic of orbit category $\mathcal{O}_G$

$$|G : N_G(H)| k_{\mathcal{O}}^H = \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|}$$

$$\sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|} = \chi(\mathcal{O}_G) = \frac{1 + (p-1)\sum |C|}{p|G|}$$

Corollary

$$|G|_{p'} |G : N_G(H)| k_{\mathcal{O}}^H \in \mathbf{Z}, \quad (1-p) \sum_{1 \leq C \leq G} |C| \equiv 1 \bmod p|G|_p$$

- The alertcategory $\mathcal{O}_G$ knows the p-radical p-subgroups

Möbius algebras

Assumption

$\mathcal{C}$ is a finite category and $[\mathcal{C}]$ is the finite set of isomorphism classes of objects of $\mathcal{C}$.

Definition (The rational Möbius algebra of $\mathcal{C}$)

$M([\mathcal{C}];\mathbf{Q})$ is the $\mathbf{Q}$-algebra with vector space basis $[\mathcal{C}]$ and

$$|\mathcal{C}(a, b \cdot c)| = |\mathcal{C}(a,b)||\mathcal{C}(a,c)|$$

Example (Burnside rings are special cases)

The Burnside ring of G is the Möbius algebra of the full orbit category $\overline{\mathcal{O}_G}$ of G. The p-Burnside ring of G is the Möbius algebra of the orbit category $\mathcal{O}_G$.

$\mathcal{C} = \mathcal{S}_G, \mathcal{T}_G, \mathcal{F}_G, \mathcal{L}_G, \widetilde{\mathcal{F}}^c_G, \ldots$ are other possibilities.

Unit, primitive idempotents, Möbius function $[\mu] = \zeta([\mathcal{C}])^{-1}$

$$1 = \sum_{[a]} [a] k_{[\mathcal{C}]}^{[a]}, \qquad e_{[b]} = \sum_{[a]} [a]\mu([\mathcal{C}])([a],[b])$$

The product $K_1 \cdot K_2$ in the Möbius algebra $M([\mathcal{C}]; \mathbf{Q})$ is

$$\begin{array}{c|c}
\mathcal{S}_G & K_1 \cap K_2 \\
\mathcal{T}_G & \displaystyle\sum_{g \in G} [K_1^g \cap K_2] \\
\mathcal{F}_G & \displaystyle\frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{F}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in [H, K_1^{g_1} \cap K_2^{g_2}]} \frac{\mu(H,K)}{|C_G(K)|} \\
\mathcal{L}_G & \displaystyle\frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{F}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in [H, K_1^{g_1} \cap K_2^{g_2}]} \frac{\mu(H,K)}{|O^pC_G(K)|} \\
\widetilde{\mathcal{F}}_G^c & \displaystyle\sum_{g \in K_1 O^p C_G(K_1) \backslash G / K_2 O^p C_G(K_2)} [K_1^g \cap K_2]
\end{array}$$

Corollary (Integral product for Möbius algebras)

Multiplication in the rational Möbius algebra $M([\mathcal{C}];\mathbf{Q})$ *restricts*

$M([\mathcal{C}];\mathbf{Z}) \times M([\mathcal{C}];\mathbf{Z}) \to M([\mathcal{C}];\mathbf{Z}), \quad \mathcal{C} = \mathcal{S}_G, \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G, \mathcal{O}_G^c, \widetilde{\mathcal{F}}_G^c$

Corollary

The p-local Möbius algebras $M([\mathcal{C}];\mathbf{Z}_{(p)})$, $\mathcal{C} = \mathcal{O}_G, \mathcal{O}_G^c, \widetilde{\mathcal{F}}_G^c$, *are commutative unital algebras.*

Theorem (Diaz–Libman 2009 [1])

There is an isomorphism of algebras

$$\varphi([\mathcal{O}_G^c],[\widetilde{\mathcal{F}}_G^c])\colon M([\mathcal{O}_G^c];\mathbf{Z}_{(p)}) \xrightarrow{\cong} M([\widetilde{\mathcal{F}}_G^c];\mathbf{Z}_{(p)})$$

given by an upper triangular nonnegative integral matrix.

Example ($M([\mathcal{F}_G; \mathbf{Q})$ for $G = \mathrm{SL}_2(\mathbf{F}_3)$, $|G| = 24$, at $p = 2$)

$M([\mathcal{F}_G]; \mathbf{Q})$	H_1	H_2	H_3	H_4
H_1	H_1	H_1	H_1	H_1
H_2	$\cdot$	H_2	H_2	H_2
H_3	$\cdot$	$\cdot$	$-H_2 + 2H_3$	$-5H_2 + 6H_3$
H_4	$\cdot$	$\cdot$	$\cdot$	$7H_2 - 18H_3 + 12H_4$

Note that coefficient sum always equals 1.

$M([\mathcal{F}_G^*]; \mathbf{Q})$	H_2	H_3	H_4
H_2	H_2	H_2	H_2
H_3	$\cdot$	$-H_2 + 2H_3$	$-5H_2 + 6H_3$
H_4	$\cdot$	$\cdot$	$7H_2 - 18H_3 + 12H_4$

$1 = \frac{2}{3}H_2 + \frac{1}{4}H_3 + \frac{1}{12}H_4$ and the Euler characteristic $\chi(\mathcal{F}_G^*) = 1$

References

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