Classifying Spaces and Cohomology of Finite Groups

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1. Homological Algebra

- Let $G$ be a group, $k$ a commutative ring of coefficients (the phrase “of coefficients” here has the empty meaning as usual)

**Definition**

$$H^*(G, k) = \text{Ext}^*_\mathbb{Z}G(\mathbb{Z}, k) \cong \text{Ext}^*_kG(k, k)$$

In other words, take a Projective Resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

and form the complex

$$0 \to \text{Hom}_{\mathbb{Z}G}(P_0, k) \to \text{Hom}_{\mathbb{Z}G}(P_1, k) \to \cdots$$

Now take homology: $\text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, k)$ is kernel mod image in $n$th place

- The answer is independent of choice of projective resolution (up to natural isomorphism)
- More generally, if $M$ is a $\mathbb{Z}G$-module we can take $\text{Hom}_{\mathbb{Z}G}(P_*, M)$ and define $H^*(G, M)$ the same way.
1. Homological Algebra, Contd.

If we tensor with $k$:

$$
\cdots \to k \otimes_{\mathbb{Z}} P_n \to \cdots \to k \otimes_{\mathbb{Z}} P_1 \to k \otimes_{\mathbb{Z}} P_0 \to k \to 0
$$

this is a projective resolution of $k$. Hence if $M$ is a $kG$-module

$$\text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M) \cong \text{Ext}^n_{kG}(k, M).$$

Some Facts:

- $H^*(G, k)$ is a graded commutative ring:
  $$yx = (-1)^{|x||y|}xy.$$

- $H^*(G, M)$ is a graded $H^*(G, k)$-module.

- (Evens) If $G$ is finite and $M$ is a Noetherian $kG$-module then $H^*(G, M)$ is a Noetherian $H^*(G, k)$-module.
Homological Algebra, Contd.

- (Evens) If $G$ is finite and $M$ is a Noetherian $kG$-module then $H^*(G, M)$ is a Noetherian $H^*(G, k)$-module.
- In particular, if $k$ is Noetherian so is $H^*(G, k)$.
- If $k$ is a field, then $H^*(G, k)$ is a finitely generated graded commutative $k$-algebra.
- If $\text{char}(k)$ is zero or does not divide $|G|$ then there’s nothing interesting here: you just get $k$ in degree zero.
- More generally, for any $k$, $|G|$ annihilates positive degree elements.

Example (The Mathieu Group $M_{11}$)

$G = M_{11}, \text{char}(k) = 2$: $H^*(G, k) = k[x, y, z]/(x^2y + z^2)$ where $|x| = 3, |y| = 4, |z| = 5.$
Commutative algebraists usually write their theorems assuming that commutative means $xy = yx$; this is bad for us.

They also often require their generators to be in the same degree; this is almost never the case for group cohomology.

Nonetheless, we can talk about the usual commutative algebra concepts such as

- Depth
- Cohen–Macaulay
- Gorenstein
- Complete Intersection
- Local Cohomology
- Castelnuovo–Mumford regularity
- etc.
2. Commutative Algebra, Contd.

**Theorem (Quillen 1971)**

If $\text{char}(k) = p$ then the **Krull dimension** of $H^*(G, k)$ is equal to the **p-rank** of $G$, namely the largest $r$ for which $(\mathbb{Z}/p)^r \leq G$.

More generally, he described the prime ideal spectrum of $H^*(G, k)$ in terms of the **elementary abelian** subgroups:

\[ H^*(G, k) \rightarrow \lim \leftarrow H^*(E, k) \]

is an **$F$-isomorphism** — it induces an isomorphism of varieties.

**Theorem (Duflot 1981)**

The depth of $H^*(G, k)$ is at least the $p$-rank of the centre of a **Sylow** $p$-subgroup of $G$. 
2. Commutative Algebra, Contd.

**Theorem (B–Carlson, 1994)**

(i) If $H^*(G, k)$ is Cohen–Macaulay then it’s Gorenstein.

(ii) If $H^*(G, k)$ is a polynomial ring then the generators are all in degree one; in this case $p = 2$ and $G$ modulo an odd order normal subgroup is $(\mathbb{Z}/2)^r$.

**Theorem (Conjectured by me in 2004, proved by Symonds 2010)**

The Castelnuovo–Mumford regularity of $H^*(G, k)$ is always equal to zero.

As a consequence, dim $H^n(G, k)$ is polynomial on residue classes, not just eventually so.
3. **Topology**

- \( EG \) - a contractible space on which \( G \) acts freely
- \( BG \) - the quotient \( EG/G \)
- \( H^*(G, k) = H^*(BG; k) \)
- Up to homotopy, \( BG \) is independent of choice of \( EG \).
- Relationship with algebraic definition:
  - \( C_*(EG) \) is an acyclic complex of free \( \mathbb{Z}G \)-modules; i.e., a free resolution of \( \mathbb{Z} \) as a \( \mathbb{Z}G \)-module.

\[
H^*(BG; k) = H^*(\text{Hom}_{\mathbb{Z}}(C_*(BG), k)) = H^*(\text{Hom}_{\mathbb{Z}G}(C_*(EG), k)) \cong \text{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, k).
\]
3. **Topology: Examples**

1. \( G = \mathbb{Z}; \ EG = \mathbb{R}; \ BG = \mathbb{R}/\mathbb{Z} = S^1. \)
   \( H^0(\mathbb{Z}, k) \cong H^1(\mathbb{Z}, k) \cong k, \ H^i(\mathbb{Z}, k) = 0 \) for \( i \geq 2. \)

2. \( G = \mathbb{Z}/2; \ EG = S^\infty; \ BG = \mathbb{R}P^\infty. \)
   If \( \text{char}(k) = 2 \) then \( H^*(G, k) = k[x] \) with \( |x| = 1. \)

3. \( G = \mathbb{Z}/2 \times \mathbb{Z}/2; \ EG = S^\infty \times S^\infty; \ BG = \mathbb{R}P^\infty \times \mathbb{R}P^\infty. \)
   If \( \text{char}(k) = 2 \) then \( H^*(G, k) = k[x, y] \) with \( |x| = |y| = 1. \)

4. \( G = Q_8 \subseteq SU(2) \cong S^3 = \text{unit quaternions} \)
   \( G \) acts freely on \( S^3 \) by left multiplication

   Cellular chains \( C_*(S^3): \)
   
   \[
   0 \rightarrow \mathbb{Z} \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0
   \]

   Form an infinite splice:
   
   \[
   \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0
   \]

   \[
   \mathbb{Z} \quad 0
   \]
Conclusion: Let $k$ be a field of characteristic 2. $H^*(Q_8, k)$ is periodic with periodicity 4. In fact the periodicity is given by multiplication by $z \in H^4(Q_8, k)$ and $H^*(Q_8, k)/(z) \cong H^*(S^3/Q_8; k)$.

**Theorem**

If $G$ acts freely on $S^{n-1}$ then $H^*(G, k)$ is periodic with period dividing $n$.

**Example**

If $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ then $G$ cannot act freely on any sphere of any dimension.
Notice that $S^3/\mathbb{Q}_8$ is a manifold so $H^*(\mathbb{Q}_8, k)/(z)$ satisfies Poincaré duality.

**Definition**

We say that a finitely generated positively graded commutative $k$-algebra is **Cohen–Macaulay** if it is a finitely generated free module over a polynomial subring $k[z_1, \ldots, z_r]$.

**Noether Normalization:** $\exists k[z_1, \ldots, z_r]$ over which it’s f.g., and whether it’s free is independent of choice of normalization.

**Theorem (B–Carlson)**

If $H^*(G, k)$ is Cohen–Macaulay then $H^*(G, k)/(z_1, \ldots, z_r)$ is a finite Poincaré duality ring with top degree $\sum_{i=1}^{r}(|z_i| - 1)$ i.e., $H^*(G, k)$ is **Gorenstein**.
More generally, even if $H^*(G, k)$ is not Cohen–Macaulay, there is a spectral sequence converging to a finite Poincaré duality ring.

Greenlees reformulated this more cleanly as a local cohomology spectral sequence

$$H_m^s H^t(G, k) \Rightarrow H_{-s-t}(G, k).$$

Symonds’ theorem states that the $E_2$ page is zero for $s + t > 0$. There is a sense in which the cochains on $BG$ are always derived Gorenstein as a DGA (Dwyer-Greenlees-Iyengar)
5. A GLIMPSE OF $p$-COMPLETION

Let $p$ be a prime. Bousfield–Kan $p$-completion is a functor from spaces to spaces together with a natural transformation $X \to X\wedge$.

- $X \to Y$ induces $H_*(X, \mathbb{F}_p) \xrightarrow{\cong} H_*(Y, \mathbb{F}_p)$ iff $X\wedge \cong Y\wedge$.
- $X$ is $p$-complete if $X\wedge \cong X\wedge$.
- $X$ is $p$-good if $X\wedge$ is $p$-complete.
- Otherwise $X$ is $p$-bad and $X\wedge \wedge \ldots$ is still $p$-bad!
- $X$ connected, $\pi_1X$ finite implies $X$ $p$-good.
- In particular if $G$ is finite $BG$ is $p$-good.
- $BG$ is $p$-complete $\iff G$ is $p$-nilpotent ($G/O_p'G$ is a $p$-group).
- The Eilenberg–Moore spectral sequence whose $E^2$ page is $\text{Tor}^{H_*(BG, \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$ doesn’t converge to $\mathbb{F}_pG$ but rather to $H_*(\Omega BG\wedge, \mathbb{F}_p)$. 
5. A Glimpse of $p$-Completion, Contd.

- $BG^\wedge_p$ only depends on the $p$-local structure of $G$.
- More precisely, there’s a category $\mathcal{L}_S^c(G)$ defined as follows:
  Let $S \in \text{Syl}_p(G)$.
- **Objects**: subgroups $H \leq S$ satisfying $C_G(H) = Z(H) \times O_{p'} C_G(H)$
- **Arrows**:
  $\text{Hom}_{\mathcal{L}_S^c(G)}(H, K) = \{ g \in G \mid gHg^{-1} \subseteq K \}/O_{p'} C_G(H)$.

**Theorem (Broto–Levi–Oliver)**

$|\mathcal{L}_S^c(G)|^\wedge_p \simeq BG^\wedge_p$, and one can recover $\mathcal{L}_S^c(G)$ from $BG^\wedge_p$. 
If $M$ is an $\mathbb{F}_p G$-module, define $[O^p G, M]$ to be the linear span of the elements $g(m) - m$ with $g \in O^p G$ and $m \in M$. This is the smallest submodule of $M$ such that the quotient has a filtration where $G$ acts trivially on the filtered quotients:

- $P_0 = N_0 = $ projective cover of $\mathbb{F}_p$ as $\mathbb{F}_p G$-module

For $i \geq 1$,
- $M_{i-1} = [O^p G, N_{i-1}]$
- $P_i = $ projective cover of $M_{i-1}$
- $N_i = \Omega M_{i-1}$

\[ \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0 \]

\[ \begin{array}{ccccc}
N_2 & \rightarrow & M_1 & \rightarrow & N_1 \\
\downarrow & & \downarrow & & \downarrow \\
M_0 & \rightarrow & N_0 & & \end{array} \]

**Theorem (B, 2009)**

\[ H_i(P_\ast) = N_i/M_i \cong H_i(\Omega BG_p^\wedge; \mathbb{F}_p). \]