Phillips Syn lecture ($C^*$-algs associated to dynamical systems)

Crossed products

$G$ a discrete group, $A$ unital $C^*$-alg

$x : G \rightarrow \text{Aut}(A)$ is an action

$C^*(G, A, x)$ is the "universal" $C^*$-alg generated

a copy of $A$ and unitaries $u_g | g \in G$ s.

satisfying $u_g a u_g^* = x_g(a)$ for $a \in A$, $g \in G$

Example (main interest)

$A = C(\mathcal{X})$ for $\mathcal{X}$ a compact space.

If $G$ acts on $\mathcal{X}$ then $x_g(f)(x) = f(g^{-1}x)$

is an action on $A$.

Example

$G = \mathbb{Z}/n$, $\mathcal{X} = \{1, 2, \ldots, n\}$ and $G$

acts on $\mathcal{X}$ by translation
Let $\mathbf{P}_n = \times_{i \leq n}$ then $\mathbf{P}_n \mathbf{P}_k = \mathbf{P}_{n+k}$ (sum is taken mod $n$).

So the unitaries $u_i$ satisfy

$$u_i \mathbf{P}_n u_i^* = \mathbf{P}_n$$

Let this act on $\mathcal{F}(\mathbf{X})$,

$$\mathbf{P}_m \mathbf{S}_n = \begin{cases} \mathbf{S}_{k} & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{P}_n u_i$ act on $\mathbf{S}_m$. First send it to $\mathbf{S}_m$,

then get $0$ if $l+m + k$ or $0$ if $l+m = k$.

Thus, these are matrix units

So the outcome is, that the crossed product is $\mathbb{M}(\mathcal{C})$.

Example

Now let $\mathbb{Z}$ act on $\{0, 1, \ldots, n-1\}$ by translation

Let $u$ be the unitary that implements the action.

Then $u^n$ is central.
One then gets a copy of
\[ c^*(n\mathbb{Z}) \cong c^*(S^1) \leq \mathbb{Z}(c^*(\mathbb{Z}, X)) \]
\[ \uparrow \]
tensor transform

So \( c^*(\mathbb{Z}, X) \) is the section of some bundle over \( S^1 \) with fibers \( M_n \). Must be trivial, since all such bundles are trivial.

Let \((X, \mu)\) be a standard probability space
\[ h: X \to X \] is an ergodic measure preserving transformation.

If \( E \subseteq X \) is \( h \)-invariant then \( \mu(E) = 0 \) or \( \mu(X \setminus E) = 0 \).

In this case there is a von Neumann algebra crossed product of \( \mathbb{Z} \) acting on \( L^\infty(X, \mu) \).
Call it \( W^*(\mathbb{Z}, X) \).
\( \text{\(c^*\)-algebra crossed product is the norm closure} \)

of (finite) sums \( \sum_{g \in \Gamma} \tau_g \otimes a_g \) w/ \( a_g \in A \).

and \( a_g = 0 \) for all but finitely many \( g \).

the \( v \)-Neumann crossed product is the

weak-topology closure of such sum.

One can show, that \( W^*(\mathbb{Z},X) \) will be a

hyperfinite II\(_1\)-factor. Since there is only one

such factor, they are all the same. So that is

why we don't really look at \( W^*(\mathbb{Z},X) \).

The \( v \)-Neumann theory is more interesting for

non-amenable gaps. At least some such gaps.
Rochlin Lemma

\[ \forall \alpha \geq 0 \exists \alpha, \forall E \in X \exists A_{n2} \rho_{n1}(E) \text{ are disjoint and} \]

\[ \mu(\{ E \setminus \bigcup_{n=0}^{\infty} \rho_{n1}(E) \}) < \varepsilon \]

Proof that \( \rho_{n2}(E, \alpha) \) is hyperfinite:

Choose \( n \gg n \), then consider the action of

a on function vanishing on \( \alpha \)

\[ E = \rho_{n1}(E) \cup \cdots \cup \rho_{n-1}(E) \]

then it looks the same as if we are acting with

\[ \sum_{k=-n}^{n} k\rho_{k}(E) \] (coming from the action of \( Z_n \))

If take \( p = \chi_E \) on \( \prod_{k \in \mathbb{Z}} \rho_{n1}(E) = S \) then \( p \rho_{n1}(E) \)

looks like something in \( \mathcal{W}^*(Z_n, S) \). Since we are using a weak topology \( \rho_{n1}(E) \) is small, so

\[ \mathcal{W}^*(Z_n, S) \sim M_{\rho_{n1}(E, \alpha)} \]

and this is hyperfinite.
Back to $C^\infty$-case.

Need $y \in \tilde{\Sigma}$ s.t. $x \in C(\tilde{\Sigma})$ for many $\Sigma$.

After finite sets the best choice is the cantor set, $h: \mathbb{R} \to \mathbb{R}$ minimal.

Pick some "small" set $Y$ compact open,

$$ (\text{will use } y_1 \supseteq y_2 \supseteq \ldots \supseteq y_0) $$

The lower line is a partition of $Y$ into compact open subsets (i.e. $y_s \in C(\tilde{\Sigma})$ for all $s$).

The whole is a partition of $\tilde{\Sigma}$ into compact open subsets.
to construct $Y$, put

$$Y = \gamma \cap h^{-n}(Y)$$

with $n$, chosen using so called "first return times".

Consider $A = C^\infty(\Xi, X, h)$

$A_Y \subset A$ is generated by $C(\Xi)$ and

$C(\Xi - Y)\hat{\otimes}$ ("magic" acting unitary) functions vanishing on $Y$.

$$A_Y \cong \bigoplus_{s \in \Gamma} M_{n_s} \left( C(\gamma_s) \right)$$

AF-algebra

$$A_{Y_0} = \bigvee A_Y \quad (\neq A \text{ but is } \text{"large"})$$

AF-algebra
Putnam has used clever tricks to get a direct limit decomposition

\[ A \sim C(\beta_0) \oplus M_3 \oplus C(S') \]

Even without using that, \( A_{\mathbb{Q},\beta} \), being a nice \( \mathfrak{c} \)-algebra can be used to get strong info about \( A \) (e.g. recent results by Sato).

Let us look at \( X \) not totally disconnected.

Take \( Y \) closed, \( \text{int}(Y) \neq \emptyset \).

\( Y_1 \) is closed

\( Y_2 \) is closed \& open

We can still form \( A_Y \). It is a sub-algebra of \( \oplus M_3 (C(Y)) \).

Suppose that there are enough invariant measures that we can expect \( A = C^\ast (\mathbb{Z}, \mathbb{Z}, h) \) to have tracial rank zero.
(One needs to get \( Y \) s.t. \( \mu(\partial Y) = 0 \) for all invariant measures) of dim \( d \).

Suppose that \( X \) is a manifold, \( \text{h} \in C^0 \).

Choose \( Y \) s.t. \( \partial Y \) is a submanifold.\( \text{distinct} \)

More over for all \( 1 \leq c \) and all \( n_1, n_2, \ldots, n_c \)

\[ h^{n_1}(\partial Y), \ldots, h^{n_c}(\partial Y) \]

are mutually transverse.

Then for \( n_1, \ldots, n_c \) \( \text{distinct} \)

\[ h^{n_1}(\partial Y) \cap \cdots \cap h^{n_c}(\partial Y) = \emptyset \]

Now for \( f = \sum_{n} x_n h^{n}(\partial Y) \) we have \( 0 \leq f \leq c \) and

\[ d = \int_{\partial Y} d\mu = \sum_{n} x_n d\mu(\partial Y) \]

so \( \mu(\partial Y) = 0 \).