What is flow equivalence of SFTs and sofic shifts (and why)?

DEFN $X$ = compact metrizable space

Flow $\varphi$ on $X$ is a continuous $R$-action,

$\varphi: X \times R \to X$

$(x, t) \mapsto \varphi_t(x)$

DEFN $C$ is a cross-section to $\varphi$ if $C$ is compact,

$C \subset X$ and $\varphi: C \times R \to X$ is a surjective local homeomorphism

($\Rightarrow$) $\varphi$ is a "flow under function" over $C$:

This is an abstraction of an early approach to studying solns. of an autonomous ODE:

Not every $\varphi$ admits a cross section.

If $X$ is one-dimensional, then $\varphi$ has a cross section.
Let $T, T'$ be self-homeomorphisms of compact metrizable spaces $X, X'$.

DEFN $T$ and $T'$ are flow equivalent if they are sections to a common flow.

And now (for better or worse) the abstract classification viewpoint:

given some class of systems of interest

and $T, T' \in \mathcal{C}$: when are $T, T'$ FE?

DEFN For $T : X \to X$, $X$ zero-dim.

a discrete cross section for $T$ is a clopen set $K$ such that all points are mapped within bounded time to $K$ by $T$ (and $T^{-1}$).

Let $T_K$ represent return map to discrete cross section $K$.

Parry-Sullivan: For all homeos $T, T'$ of zero-dim compact metr. $X$ spaces:

$T \equiv T' \iff \exists K, K' \ni T_K = T'_K$.

DEFN OF SET $\sigma_A : X_A \to X_A$

Ex. $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

$G = \vdots^n \begin{pmatrix} c \\ d \end{pmatrix}$

$E = \{a, b, c, d\}$

$X_A = \{x = \ldots x_n x_0 x_1 \ldots : x_n \in E, x_{n+1} \text{ follows } x_n \}$

$(\sigma_A x)^n = x^{n+1}$

$\sigma_A : X_A \to X_A$ homeo.

$\sigma_A$ is irreducible if $A$ is irreducible.
Before results on FE — consider "why".

(1) Interaction with/utility for C*-classification
work on Cuntz-Krieger and graph C*-algebras
e.g. — Rordam's stable classification and
Restorff's of C*-algebras (using Huang)
- 2013 Matsumoto-Matui
  (using B-Handelman)

(2) An important part of a general classification
structure for SFTs and related systems

**Theorems** For SFT \((X_A, \sigma_A)\):

- Parry-Sullivan Theorem \(\overset{P-S}{\Rightarrow} \det(I-A)\) is an invariant of FE
- Bowen-Franks \(\overset{B-F}{\Rightarrow} \text{Cok}(I-A)\)

Irreducible \(X_A\) (not just a single orbit):
- Franks: \((i) + (ii) = \text{complete invariant of FE}\)

General \(X_A\):
- Huang, B-Huang: \text{complete alg. invariants}
  (complicated "K-web")

Classification of Cuntz-Krieger algebras up to
stable isomorphism (Rordam; Restorff) using
FE-classification of SFTs (Franks; Huang)
Glimpse of general classification framework

Polynomial Matrix \( \rightarrow \Phi \) (Shannon, BOMT, KRW, BW)

Ex.

\[
A = \begin{bmatrix} t^2 & t \\ 2t & 0 \end{bmatrix}
\]

over \( \mathbb{Z}_+ [t] \)

\( A^\# = \text{adj. matrix} \) \( \Rightarrow \text{disc. c.s.} \)

\( \text{SPT} X_A^\# = X_A \)

Polynomial matrix can be a concise presentation of complicated graph (SFT). But it is more.

Note: changing pos. factors of \( t \) does not change PE class of \( X_A \)

Consider \((I - A): (\mathbb{Z}[t])^n \otimes \mathbb{Z}^n \quad v \mapsto Av \)

\( \mathbb{Z}[t] \)-module \( \text{cok}(I - \otimes A(t)) \): strong invariant of \( \mathbb{Z} \) (top. conjugacy) for SFT

\( + \rightarrow 1 \) \( \downarrow \)

\( \mathbb{Z} \)-module \( \text{cok}(I - A(1)) \): strong invariant of PE

(We don't see this functionality if we restrict to \( \mathbb{Z}_+ \)-matrices.) \( \sigma_A \equiv \sigma_B \Leftrightarrow A \sim_{\mathbb{Z}_+} B \)

(The equivalence relation \( \sim_{\mathbb{Z}_+} \) on square matrices over \( \mathbb{Z}_+ \) is the transitive closure of \( RS \sim SR \) for matrices \( R,S \) over \( \mathbb{Z}_+ \).)

(As operations on \( \mathbb{Z} \)-matrices - it is not clear that \( \sim_{\mathbb{Z}} \) and \( \text{cok}(I - A) \) are related.)
A statement of the framework (for SFTs $X_A$, $A$ over $\mathbb{Z}_+[t]$) (neglecting any technicality):

$$A \sim (I-A \circ \begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix}) = (I-A)_a$$

$m \geq n$

For $A, B$ over $\mathbb{Z}_+[t]$:
say $A \sim B$ if the matrix $E_{i,j}(t^k)$, and
$$E(I-A) = I-B$$
$$\sim (I-A)E = I-B$$

Say $A \Leftrightarrow B$ if $A \sim A, \sim A \sim B$.

Then

SFTs $\sigma_A, \sigma_B$ are $\Leftrightarrow$

$\Rightarrow A \sim B$.

Looks like $K_1(\mathbb{Z}_+[t])$, except

(i) $I-A$ is not invertible

(ii) the chain for $\sim$ requires every $A_i$ over $\mathbb{Z}_+[t]$

It is not known if there is a decision procedure.

For (i) For $\mathbb{Z}_+[t]$ and general $R[\pm 1]$, we know what this means algebraically

(ii) Still a mystery.

It is not known if $\exists$ decision procedure for $\sigma_A \not\sim \sigma_B$. 

\[ \square \]
Sofic shifts arise from labelled directed graphs.

\[ X = \{ x = \cdots x_{-1} x_0 x_1 \cdots \}_{x \in \{0,1\}^\mathbb{Z}} \]

forbidden words: \( 10^{2n+1} \)

"even shift"

FE of sofic shifts — "natural" question

- no external connection yet (e.g. with C*-algebras)
- \( \mathbb{G} \)
- far more complicated than FE of SFTs

**DEFN** Let \( G \) be a finite group. Then a free \( G \)-SFT is an SFT with cont. \( G \)-action commuting with the shift.

A free \( G \)-SFT can be presented by a matrix \( A \) over \( \mathbb{Z} \times G \). The "positive K-theory" framework holds for these systems using the ring \( \mathbb{Z} \times G \) and positive set \( \mathbb{Z}^+ \times G \).

**THM** (B-Sullivan) For mixing \( G \)-SFTs \( \sigma_A \), \( \sigma_B \):

\[ \sigma_A \text{ G-FE } \sigma_B \iff (I-A) \alpha \parallel (I-B) \alpha \]

and in the class of matrices \( \GL(\mathbb{Z}^G) \)-equiv. to \( [I-A] \):

the G-FE classification is given by

\[ K_1(\mathbb{Z}^G)/H_A \]

where \( H_A = \{ [U] : EE \in \mathbb{E}(\mathbb{Z}^G) : U(I-A)E = IAG \} \)
Surprisingly (?),

$G$-FE of $G$-SFTs is very relevant to FE of Sofic shifts.

In the Center for Symmetry and Deformation, we could see it as an example of an appearance of unexpected symmetry.