Index and determinant of commuting operators

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Basic data: n-tuple $A = (A_1, \ldots, A_n)$ of commuting operators on a Hilbert space $\mathcal{H}$.

Generalization of the notion of spectrum and holomorphic calculus of a single operator (Taylor):

1. $K(A)$ - the Koszul complex of $A$: $(\mathcal{H} \otimes \Lambda^* \mathbb{C}^n, d_A)$ where the differential is given by $d_A = \sum_i A_i \otimes \iota e_i$, $((e_1, \ldots, e_n)$ the standard basis of $\mathbb{C}^n$.

2. Spectrum of $A$: $\sigma(A) = \{ \lambda \in \mathbb{C}^n \mid K(A - \lambda) \text{ is not acyclic} \}$

3. Fredholm spectrum of $A$: $\sigma_F(A) = \{ \lambda \in \sigma(A) \mid d_{A - \lambda} \text{ is Fredholm} \}$

4. Essential spectrum of $A$:
   $\sigma_{ess}(A) = \{ \lambda \in \sigma(A) \mid d_{A - \lambda} \text{ is not Fredholm} \}$

5. Multivariable holomorph functional calculus:
   $\phi_A : \mathcal{O}_A \to B(\mathcal{H})$
   where $\mathcal{O}_A$ is the ring of germs of functions holomorphic in a neighbourhood of $\sigma(A)$ and, for $f$ holomorphic in a polydisc containing $\sigma(A)$, $\phi_A(f) = f(A_1, \ldots, A_n)$.
Suppose we are given \( g = (g_1, \ldots, g_n) \in O^n_A \). Then \( (g_1(A), g_2(A), \ldots, g_n(A)) \) is again an \( n \)-tuple of commuting bounded operators.

Questions:

- When is \((K(g(A)), d_g(A))\) Fredholm

Given that \((K(g(A)), d_g(A))\) is Fredholm,

1. Compute its Fredholm index in local terms (explanation later)
2. Study the determinant line \( \text{Det}(d_g(A)) \).
3. Suppose that \( f \in O^{n-1}_A, \ g, h \in O_A \) are such that \( K(f(A), g(A)) \) and \( K(f(A), h(A)) \) are Fredholm. Compute the quotient \( \text{Det}(d(f(A), g(A))) / \text{Det}(d(f(A), h(A))) \).
Simple example to keep in mind - Toeplitz operators

1. \( \mathcal{H} = H^2 \), the Hilbert space of holomorphic \( L^2 \)-functions on the open unit disc \( \mathbb{D} \). \( A = T_z \), the multiplication operator by the complex coordinate \( z \).

2. \( \sigma(T_z) = \overline{\mathbb{D}} \) and \( \sigma_F(T_z) = \mathbb{C} \setminus \partial \mathbb{D} \).

3. \( f(T_z) = T_f \) is Fredholm iff \( \{ f(z) = 0 \} \cap \partial \mathbb{D} = \emptyset \)

4. Suppose that \( T_f \) is Fredholm. Then (Fritz Noether)

   \[
   \text{Index} T_f = - \sum_{\lambda \in \{ f(z) = 0 \} \cap \mathbb{D}} \deg_{\lambda} f
   \]

   where \( \deg_{\lambda} f \) stands for the multiplicity of the zero of \( f \) at \( \lambda \).

5. Suppose that \( T_f \) and \( T_g \) are Fredholm. Then (Carey, Pincus)

   \[
   \det(T_f)/\det(T_g) = (-1)^{\text{Index}(T_f)\text{Index}(T_g)} \frac{\prod_{g(\lambda) = 0} f(\lambda)^{\deg_{\lambda} g}}{\prod_{f(\mu) = 0} g(\mu)^{\deg_{\mu} f}}
   \]
Let us start with the global results. Given an n-tuple \( g \in \mathcal{O}_A^m \)

**Theorem (Eschmeier, Putinar, Levy)**

1. \( g(A) \) is Fredholm iff the set \( g^{-1}(\{0\}) \cap \sigma_{ess}(A) = \emptyset \).
2. Suppose that \( g(A) \) is Fredholm. Then \( g^{-1}(\{0\}) \cap \sigma_F(A) \) is finite and

\[
\text{Index}(g(A)) = \sum_{\lambda \in Z(g)} \deg_\lambda(g) \text{Index}(A - \lambda)
\]

where \( \deg_\lambda(g) \) is the intersection multiplicity of \( \{g_1 = \ldots = g_n = 0\} \) at \( \lambda \).
Localization

The holomorphic functional calculus gives the Hilbert space $\mathcal{H}$ the structure of a $\mathcal{O}_A$-module. For each $\lambda \in \sigma(A)$ set

$$p_\lambda = \{ f \in \mathcal{O} \mid f(\lambda) = 0 \}$$

The Koszul complex of $A$ localizes, hence

**Definition**

The local index at $\lambda$ is

$$\text{Index}_\lambda(g(A)) := - \sum_i (-1)^i \text{Dim}_\mathbb{C}(H_i(g, \mathcal{H}_{p_\lambda})).$$

Here $H_*(g, \mathcal{H}_{p_\lambda})$ stand for the homology groups of Koszul complex $K(g(A))$ localised at $p_\lambda$. Note that the homology groups are finite dimensional as a consequence of the Fredholmness of $g(A)$.

Easy consequence of the definition:

$$\text{Ind}(g(A)) = \sum_{\lambda \in \sigma(g)} \text{Ind}_\lambda(g(A)).$$
Suppose that \( g(A) \) Fredholm.

The localized homology groups \( H_*(g, \mathcal{H}_{p,\lambda}) \) can be turned into a graded module over the ring \( \mathcal{O}_\lambda \) of power series convergent near \( \lambda \in Sp(A) \). The associated homomorphism \( \mathcal{O}_\lambda \to \text{End}(H(g(A), \mathcal{H}_{p,\lambda})) \) makes the diagram

\[
\begin{array}{ccc}
\mathcal{O}_A & \longrightarrow & \mathcal{O}_\lambda \\
\downarrow & & \downarrow \\
\text{End}(H(g(A), \mathcal{H}_{p,\lambda})) & \longrightarrow & \text{End}(H(g(A), \mathcal{H}_{p,\lambda}))
\end{array}
\]

commute. In particular, the classical Lie theorem applied to the commuting n-tuple \( H_*(A_1), \ldots, H_*(A_n) \) on \( H(g(A), \mathcal{H}) \) gives

\[
H(g(A), \mathcal{H}) = \bigoplus_{\lambda} H(g(A), \mathcal{H})(\lambda)
\]

and, for any \( \phi \in \mathcal{O}_A \), the action of \( \phi(A) \) on \( H(g(A), \mathcal{H})(\lambda) \) has the form of an upper triangular matrix with \( \phi(\lambda) \) on the diagonal.
Local index theorem can now be stated as follows.

**Theorem**

Suppose that $g(A)$ is Fredholm and that $g(\lambda) = 0$. Then the homology groups $H_* (g, \mathcal{H}_\lambda)$ are finite dimensional and

$$Ind_\lambda (g(A)) = deg_\lambda (g) Ind(A - \lambda),$$

where $deg_\lambda (g)$ is the intersection multiplicity of $\{ g_1 = \ldots = g_n = 0 \}$ at $\lambda$. 
Determinants

Given a Fredholm operator $T : H_+ \to H_-$, the corresponding determinant line is the one dimensional complex vector space

$$Det T = \Lambda^{top}(\text{Ker} T) \otimes \Lambda^{top}(\text{Coker} T)^*$$

Let us recall some basic facts.

1. $T \to Det(T)$ is a continuous line bundle over the spaces of Fredholm operators (in fact analytic in the case of $H_+ = H_-$)

2. Suppose that

$$S : 0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow 0$$

is an exact complex of finite dimensional vector spaces. Then there exists a natural isomorphism

$$Det(S) : Det(V_1) \to Det(V_0) \otimes Det(V_2).$$
We need to formalise it a bit. The Picard category $\mathcal{P}ic$ of graded lines is the following.

- Objects of $\mathcal{P}ic$ are pairs $(L, n)$ where $L$ is a one dimensional vector space (line) and $n \in \mathbb{Z}$
- $\text{Mor}_{\mathcal{P}ic}$ is the set of zero degree invertible linear transformations.

$\mathcal{P}ic$ forms a symmetric monoidal category with inverse:

- $(L_1, n_1) \otimes (L_2, n_2) = (L_1 \otimes L_2, n_1 + n_2)$;
- The isomorphism $(L_1, n_1) \otimes (L_2, n_2) \rightarrow (L_2, n_2) \otimes (L_1, n_1)$ is given by
  $$\xi \otimes \eta \rightarrow (-1)^{n_1n_2} \eta \otimes \xi;$$
- The unit element is $(\mathbb{C}, 0)$ and the left inverse is given by involution
  $$(L, n) \rightarrow (L, n)^\dagger = (L^*, -n).$$
- Given a vector space $V[-n]$ (i.e. placed in degree $n$), set
  $$\text{Det}V = \begin{cases} 
    (\Lambda^{\text{top}} V, \text{dim}V) & n \equiv 0(2) \\
    (\Lambda^{\text{top}} V, \text{dim}V)^\dagger & n \equiv 1(2)
  \end{cases}$$

and $\text{Det}(0) = (\mathbb{C}, 0)$
Let $\mathcal{V}ect$ be the category of bounded complexes of finite dimensional vector spaces and $\mathcal{V}ect_0$ the subcategory with the same objects but with morphisms being the quasi-isomorphisms of complexes. The determinant functor

$$Det : \mathcal{V}ect_0 \rightarrow \mathcal{P}ic$$

is given by

$$C \mapsto \cdots \otimes Det(H_i(C)) \otimes Det(H_{i+1}(C)) \otimes \cdots$$

The same construction works in the category $\mathcal{F}red$ of finite Fredholm complexes, i.e. of the form

$$C = \cdots \rightarrow H_k \xrightarrow{d_k} H_{k-1} \xrightarrow{d_{k-1}} H_{k-2} \xrightarrow{d_{k-2}} \cdots$$

where, for each $k$, $Rg(d_k)$ is closed of finite codimension in $Ker(d_{k-1})$. 
\[ \mathcal{F}_{\text{red}} \text{ is a triangulated category, with} \]

- \[ C \to C[1] \] (shift the grading to the left)
- and exact triangles: given a morphism \[ f : A \to B \] in \( \mathcal{F}_{\text{red}} \) there is a mapping cone \( C_f \), the total complex of the double complex

\[
\begin{array}{c}
\vdots \\
\ldots \to A_k \to A_{k-1} \to A \to \ldots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\ldots \to B_k \to B_{k-1} \to B \to \ldots \\
\end{array}
\]

which fits into an exact triangle

\[ \Delta_f : C_f[-1] \to A \xrightarrow{f} B \xrightarrow{-f} C_f \]

Exact triangle as above induces a natural isomorphism

\[ \text{Det}(\Delta_f) : \text{Det}(C_f) \to \text{Det}(B) \otimes \text{Det}(A)[1]. \]
Some examples for later.

1) Suppose that $\mathcal{A} = \mathcal{B}$. Then $\text{Det}(\Delta)$ is the trivialization of $\text{Det}(\mathcal{C}_f)$ given by

$$
\begin{array}{c}
\text{Det}(\mathcal{C}_f) \xrightarrow{\text{Det}(\Delta)} \text{Det}\mathcal{A} \otimes \text{Det}\mathcal{A}[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{Det}(\mathcal{A}) \otimes \text{Det}(\mathcal{A})^\dagger \\
\downarrow \quad \downarrow \\
\mathbb{C}
\end{array}
$$

2) Suppose moreover $f$ is a quasi-isomorphism, i.e. $H_*(\mathcal{C}_f) = 0$. Then the composition

$$
\begin{array}{c}
\mathbb{C} = \text{Det}(\mathcal{C}_f) \xrightarrow{\text{Det}(\Delta)} \text{Det}\mathcal{A} \otimes \text{Det}\mathcal{A}[1] = \mathbb{C}
\end{array}
$$

given (up to a sign) by the multiplication with

$$
det(f) = \prod_n det(f_{2n})det(f_{2n+1})^{-1}
$$
Back to our Koszul complexes. Let \( h = (h_1, \ldots, h_{n-1}) \in \mathcal{O}_A^{n-1} \) and \( f, g \in \mathcal{O}_A \) Then

The Koszul complex \( K(h(A), f(A), g(A)) \) coincides with both

1. the mapping cone \( C_f \) of

\[
    f(A) : K(h(A), g(A)) \to K(h(A), g(A))
\]

and

2. the mapping cone \( C_g \) of

\[
    g(A) : K(h(A), f(A)) \to K(h(A), f(A)).
\]

In particular, by above example, we get two isomorphisms

\[
    \text{Det}(\Delta_f) : \text{Det}(K(h(A), f(A), g(A))) \to \mathbb{C}
\]

and

\[
    \text{Det}(\Delta_g) : \text{Det}(K(h(A), f(A), g(A))) \to \mathbb{C}.
\]

The torsion of the \( n+1 \) tuple \((h_1, \ldots, h_{n-1}, f, g)\) is

\[
    \text{Tor}(f, g) = \text{Det}(\Delta_f)\text{Det}(\Delta_g)^{-1} \in \mathbb{C}^*
\]
Theorem

Suppose that $h_1, h_2, \ldots, f, g \in \mathcal{O}_A$ are such that

$$V(h, f) \cap \sigma_{\text{ess}}(A) = \emptyset \text{ and } V(h, g) \cap \sigma_{\text{ess}}(A) = \emptyset,$$

where $V(f, \ldots)$ stands for the common set of zero's of $(f, \ldots)$. Then

$$\text{Tor}(f, g) = \epsilon \frac{\prod_{\lambda \in V(h, g)} f(\lambda)^{\text{deg}_\lambda V(h, g) \text{Index}(A - \lambda)}}{\prod_{\mu \in V(h, f)} g(\mu)^{\text{deg}_\mu (h, f) \text{Index}(A - \mu)}}$$

where the sign $\epsilon$ is given by

$$(-1)^{\text{Ind}(K(f(A), h(A))) \text{Ind}(K(g(A), h(A)))}.$$

In the case of $\mathcal{H} = L^2_{\text{hol}}(\mathbb{D})^n$ and $A$ given by the coordinate functions $(z_1, \ldots, z_n)$ this can be interpreted as the Tate tame symbol of the $n+1$-tuple of holomorphic functions $(h_1, h_2, \ldots, f, g)$. 
Sketch of proof.

Suppose first that the $n+1$ tuple $(h_1, h_2, \ldots, f, g)$ has no common zeroes within $\sigma(A)$. Then $K(h(A), f(A), g(A))$ is contractible, hence

$$\text{Det}(\Delta_f) = \text{determinant of } \{ f(A) : \bigoplus H_*(h(A), g(A); \mathcal{H}) \}$$

Since $H_*(h(A), g(A); \mathcal{H}) = \bigoplus_{\lambda \in V(h,g)} H_*(h(A), g(A); \mathcal{H})(\lambda)$ with $f(A)$ acting on the components $H_*(h(A), g(A); \mathcal{H})(\lambda)$ as upper triangular matrices with constant diagonals $f(\lambda)$, the claim follows easily.
Sketch of proof.

Suppose first that the n+1 tuple \((h_1, h_2, \ldots, f, g)\) has no common zeroes within \(\sigma(A)\). Then \(K(h(A), f(A), g(A))\) is contractible, hence

\[
Det(\Delta_f) = \text{determinant of } \{ f(A) : H^\bullet(h(A), g(A); \mathcal{H}) \}
\]

Since \(H^\bullet(h(A), g(A); \mathcal{H}) = \bigoplus_{\lambda \in V(h, g)} H^\bullet(h(A), g(A); \mathcal{H})(\lambda)\) with \(f(A)\) acting on the components \(H^\bullet(h(A), g(A); \mathcal{H})(\lambda)\) as upper triangular matrices with constant diagonals \(f(\lambda)\), the claim follows easily.

The general case follows from the continuity of torsion. In fact, the following (non-trivial!) theorem holds.

For an open set \(U \subset \mathbb{C}^n\), let \(O_U\) denote the ring of functions holomorphic in \(U\) and extending continuously to \(\overline{U}\). We give \(O_U\) the Banach space topology with the \(\| \cdot \|_\infty\) norm. Given an n-tuple \(A\), we give the ring \(O_A\) the direct limit topology associated to the filter of open neighbourhoods of \(\sigma(A)\).

The map

\[
(f, g, h_1, \ldots, h_{n-1}) \in O_A^{n+1} \rightarrow Tor(f, g)
\]

is holomorphic.