What is Index Theory good for: obstructions to positive scalar curvature

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What are the possibilities for *geometry* on a given *topology*?

“Geometry here means: curvature features of *Riemannian* geometry.
Motivation:

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“Geometry here means: curvature features of Riemannian geometry.

“Topology” means: we fix a (smooth compact boundaryless) manifold $M$, like $S^n$, $T^n = S^1 \times \cdots \times S^1$, ...
More specifically: what are the possibilities for its \textit{scalar curvature}?
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More specifically: what are the possibilities for its *scalar curvature*?

In higher dimensions: $\text{scal}(x)$ is the average (integral) of the scalar curvature of all 2-d surfaces through the point $x$. 
Scalar curvature in higher dimension.

The scalar curvature can be computed as a trace of the curvature tensor 
\[ \text{scal} = \sum_{ij} R^j_{iji}. \] Alternatively (feature taken as def):

**Definition**

Given a Riemannian manifold \((M, g)\), scalar curvature \(\text{scal} : M \to \mathbb{R}\) satisfies

\[
\frac{\text{vol}(B_\epsilon(x) \subset M)}{\text{vol}(B_\epsilon(x) \subset \mathbb{R}^m)} = 1 - \frac{\text{scal}(x)}{6m + 2} \epsilon^2 + O(\epsilon^4).
\]
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Theorem (Gauß-Bonnet)

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Corollary: If $\text{scal} > 0$ on surface $F$ implies the Euler characteristic of $F$ is positive, i.e. $F = \mathbb{S}^2$, $\mathbb{R}P^2$. 
Positive scalar curvature and Gauß-Bonnet

**Theorem (Gauß-Bonnet)**

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**Corollary**

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Basic Dirac operators

Dirac: Differential operator $D$ as square root of matrix Laplacian (using Pauli matrices).

1D: $\Delta = -\partial_x^2$; $D = i\partial_x$

higher D: $\Delta = \sum -\partial_j^2$, need: $D = \sum \epsilon_j \partial_j$ with $\epsilon_j \epsilon_i + \epsilon_i \epsilon_j = -\delta_{ij}$.

These relations are the relations of the generators of the Clifford algebra; in 4D they are satisfied by the Pauli-Dirac matrices.

Schrödinger: generalization to curved space-time (local calculation) satisfies

$$D^2 = \nabla^* \nabla + \frac{1}{4} \text{scal}.$$
**Setting**: our compact smooth manifold $M$ without boundary

Given a spin structure (a strengthened version of orientation) and a Riemannian metric, one gets:

1. the spinor bundle $S$ over $M$. Sections of this bundle are spinors, (i.e. certain vector-valued functions, where the vector space depends smoothly on the point)

2. the Dirac operator $D$ acting on spinors: a first order differential operator which is *elliptic*. Example: In flat space, $D$ is a square root of the vector Laplacian.
Elliptic operators and index

- Analytic fact: an elliptic operator $D$ on a compact manifold is Fredholm: it has a quasi-inverse $P$ such that

$$DP - 1 = Q_1; \quad PD - q = Q_2$$

and $Q_1, Q_2$ are compact operators. The compact operators are norm limits of operators with finite dimensional image. They form an ideal in the algebra of all bounded operators.

- Consequence of Fredholm property: null-space of the operator and of the adjoint are finite dimensional, and then

$$\text{ind}(D) := \dim(\ker(D)) - \dim(\ker(D^*)) .$$

- We apply this to the Dirac operator (strictly speaking to its restriction $D^+$ to positive spinors).
If $M$ is compact spin, we have the celebrated

**Theorem (Atiyah-Singer index theorem)**

\[ \text{ind}(D) = \hat{A}(M) \]

Here $\hat{A}(M)$ is a differential-topological invariant of the manifold which can be efficiently computed without ever solving differential equations. It does not depend on the metric ($D$ does).
Characteristic classes and characteristic numbers

Classical subject:
- To vector bundle $E \to B$ we assign cohomology classes $p(E) \in H^*(B)$ (Chern classes, Pontryagin classes, (Stiefel Whitney classes))
- The are natural: $p(f^*E) = f^*p(E)$
- Sum and product formulas
- Calculation using curvature differential form (of connection on bundle)
- If $B = M$ closed manifold and $p \in H^{\dim(M)}(M)$ we get associated characteristic number $\int_M p(E) = \langle p(E), [M] \rangle$

Applied to $TM$ get characteristic classes and numbers of smooth manifolds.
Schrödinger-Lichnerowicz formula and consequences

**Theorem**

Schrödinger’s (rediscovered by Lichnerowicz) local calculation relates the Dirac operator to positive scalar curvature. It implies:

if the scalar curvature is everywhere positive, then the Dirac operator is really invertible (not only modulo compact operators) (more explicitly, one shows that $D^2$ is strictly positive, which implies invertibility of $D$).

**Proof.**

triviality of kernel:

$$Ds = 0 \implies D^2s = 0 \implies 0 = \langle D^2s, s \rangle = \langle Ds, Ds \rangle = 0 \implies Ds = 0$$

$$0 = \langle D^2s, s \rangle = \langle \nabla^* \nabla s, s \rangle + \left\langle \frac{\text{scal}}{4}, s, s \right\rangle = |\nabla s|^2 + \frac{1}{4} \int_M \text{scal}(x) \langle s(x), s(x) \rangle_x$$
Consequence: If $M$ has positive scalar curvature, then $\text{ind}(D) = 0$, therefore $\hat{A}(M) = 0$:

$\hat{A}(M) \neq 0$ is an obstruction to positive scalar curvature!

**Example**: Kummer surface $K3$ (in general: algebraic geometry of smooth varieties over $\mathbb{C}$ a great help to calculate characteristic classes).

**Non-example**: $\mathbb{C}P^2$ has $\hat{A}(\mathbb{C}P^2) = -1/8 \neq 0$, but definitely does admit positive scalar curvature (the standard Fubini-Study metric even has positive sectional curvature). It does not have a spin structure. Second application of index theorem: **Integrality** of characteristic numbers.

**Non-example**: $T^n$ — the tangent bundle has a flat connection (it is even trivial), so all characteristic classes and numbers vanish.
Slogan: Index is based on an element in the algebra of (bounded) operators which is invertible modulo the ideal of compact operators. This is encoded in K-theory: we have the exact sequence of $C^*$-algebras

$$0 \rightarrow K \rightarrow B \rightarrow B/K \rightarrow 0$$

of the ideal $K$ of compact operators inside the algebra $B$ of bounded operators, with quotient the Calkin algebra $B/K$. 
We have for a (stable) $C^*$-algebra $A$:

- $K_1(A)$ are homotopy classes of invertible elements over $A$
- $K_0(A)$ are homotopy classes of projections in $A$
- 6-term long exact K-theory sequence for ideal $I \subset A$:

$$
\rightarrow K_0(A/I) \rightarrow K_1(I) \rightarrow K_1(A) \rightarrow K_1(A/I) \xrightarrow{\text{ind}} K_0(I) \rightarrow
\rightarrow K_0(A) \rightarrow K_0(A/I) \rightarrow
$$

**Example**

An elliptic operator is invertible modulo $K$, i.e. represents an element in $K_1(B/K)$, its index is then an element in $K_0(K) = \mathbb{Z}$. 
**General goal**: find in a given situation appropriate algebras to arrive at similar index situations. Criteria:

- index construction must be possible (operator in $A$, invertible modulo an ideal $I$)
- calculation tools for $K_*(I)$
- relevant geometric conditions which imply vanishing of index
- Useful/crucial is the context of $C^*$-algebras, where *positivity implies invertibility.*
Non-compact manifolds

What can we do if $M$ is not compact?
Why care in the first place?
Non-compact manifolds

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Why care in the first place?
This is of relevance even when studying compact manifolds:

- extra information can be obtained by studying the covering spaces with their group of deck transformation symmetries (e.g. $\mathbb{R}^n \to T^n$ with deck transformation action by $\mathbb{Z}^n$).
- attaching an infinite half-cylinder to the boundary of a compact manifold with boundary assigns a manifold without boundary, but which is non-compact.
- Interesting in their own right.
There are dedicated $C^\ast$-algebras (of bounded operators) adapted to the analytic features of Dirac operators:

$$0 \to C^\ast(M)\Gamma \to D^\ast(M)\Gamma \to D^\ast(M)\Gamma / C^\ast(M)\Gamma \to 0$$

such that the Dirac operator (if $\Gamma$ acts isometrically on the complete manifold $M$) defines an operator in $D^\ast(M)\Gamma$ invertible module $C^\ast(M)\Gamma$, therefore a class in $K_\ast(D^\ast(M)\Gamma / C^\ast(M)\Gamma)$, but if one has uniformly positive scalar curvature, it is even invertible in $D^\ast(M)\Gamma$ and therefore one has a lift to $K_\ast(D^\ast(M)\Gamma)$ (the lift depends on the psc metric). Therefore, the index in $K_\ast(C^\ast(M)\Gamma$ is an obstruction to psc on $M$.

Homotopy invariance: the class doesn’t change if one deforms the metric within its bilipschitz class.

**multipartitioned manifold index theorem**: reduces to compact submanifold. Works e.g. for $\mathbb{R}^n$ and therefore the torus.
Theorem

Assume $N \subset M$ is a codimension $k$ submanifold with trivial normal bundle (both closed). Assume $\pi_1(N) \to \pi_1(M)$ is injective. Assume $\pi_j(M) = 0$ for $2 \leq j < k$ and $\pi_k(N) \to \pi_k(M)$.

Assume that the Mishchenko index $\text{ind}(D_N) \in KO^*(C^*\pi_1N)$ is non-zero.

Then $M$ does not have positive scalar curvature if

1. $n = 1$ (Rudi Zeidler) —full picture including transfer
2. $n = 2$ (Hanke-Pape-Schick) —$C^*$-proof
3. general $n$, if the non-vanishing holds rationally and the Novikov conjecture (rational injectivity of Baum-Connes assembly map) holds for $\pi_1M$ —topological transfer proof (Engel, Schick-Zeidler)

Higson-Schick-Xie: generalization to homotopy invariance of codimension 2 signatures
Coarse geometry is focusing on the *large scale features* of a (metric) space.

**Definition**

Two metric spaces $X$, $Y$ are coarsely equivalent if there is a maps $f : X \rightarrow Y$, $g : Y \rightarrow X$ such that

- $f, g$ are coarse maps, i.e. $\forall c > 0$ there is $C > 0$ such that if $d(x, y) < c$ then $d(f(x), f(y)) < C$, and the inverse image of every bounded set is bounded.
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- both $f \circ g$ and $g \circ f$ are close to the identity, i.e. $\exists C$ such that $d(f \circ g(x), x) \leq C$ for all $x$.

Note that $f, g$ are not required to be continuous.
$\mathbb{Z}^n$ and $\mathbb{R}^n$ are coarsely equivalent, with $f$ the inclusion and $g$ the integer part.

**Slogan:** the coarse type is what one sees if one looks at a space from very far away.

- In general, if a discrete group $\Gamma$ acts freely isometrically on a metric space $X$ with compact quotient $X/\Gamma$, then $X$ and the orbit of any point (corresponding to $\Gamma$) are coarsely equivalent.
- All compact metric spaces are coarsely equivalent to each other.
Coarse $C^*$-algebras

**Definition**

$M$ a Riemannian spin manifold (not necessarily compact). The coarse algebra/Roe algebra $C^*(M)$ is the algebra of bounded operators $T$ on $L^2(S)$ satisfying

- $T$ has finite propagation: there is $R_T$ such that the support of $T(s)$ is contained in the $R_T$-neighborhood of the support of $s$ for each $s$. 

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- $T$ has finite propagation: there is $R_T$ such that the support of $T(s)$ is contained in the $R_T$-neighborhood of the support of $s$ for each $s$.
- Local compactness: if $\phi \in C(M)$ has compact support, the composition of $T$ with multiplication by $\phi$ (on either side) is a compact operator.
- Picture: the Schwarz (pseudodifferential integral) kernel of $T$ has support in an $R_T$-neighborhood of the diagonal, and each “bounded piece” of it is a compact operator.
Example

If $M$ is compact, $C^*(M)$ is the ideal of compact operators.

Theorem

The isomorphism type of $C^*(M)$ depends only on the coarse type of $M$. Coarse maps induce canonical and functorial $C^*$-algebra homomorphisms between the Roe algebras.
Not only coarse $C^*$-algebras

**Definition**

The algebra $D^*(M)$ is the “normalizer” of $C^*(M)$ in $B(L^2(M))$: the largest subalgebra such that $C^*(M)$ is an ideal in $D^*(M)$ (also called the multiplier algebra).

**Lemma**

$C^*(M)$ is an ideal of $D^*(M)$. $D^*(M)$ (and its K-theory) do depend on the small scale topology of $M$.

(with $D^*(M)$ we are slightly deviating from the usual notation.)
The Dirac operator on a spin manifold $M$ defines an invertible element $\chi(D) \in D^*(M)/C^*(M)$ and therefore a coarse index

$$\text{ind}_c(D) \in K_*(C^*(M))$$

Even better:

**Theorem**

*If an interval around 0 is not in the spectrum of $D$ (i.e. $D^2$ is strictly positive, which follows from positive scalar curvature), the Dirac operator again defines an element which is invertible in $D^*(M)$, so $\text{ind}_c(D) = 0 \in K_*(C^*(M))$ in this case.*

(Again, we really have to use the Dirac operator restricted to positive spinors)
There are good tools to compute $K_\ast(C^\ast(M))$, e.g. a

- *Mayer-Vietoris sequence* to put the information together by breaking up $M$ in simpler pieces
- vanishing results for suitable kinds of coarse contractibility, in particular if $X = Y \times [0, \infty)$.

Example consequence:

**Theorem**

$$K_0(C^\ast(\mathbb{R}^{2n})) = \mathbb{Z}; \quad K_1(C(\mathbb{R}^{2n+1})) = \mathbb{Z}.$$
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**Theorem (Partitioned manifold index theorem)**

$p_*(\text{ind}_c) = \hat{A}(P)$.

**Corollary**

$\tilde{M}$ and therefore $M$ does not admit a metric of positive scalar curvature.
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Theorem

If $M$ contains a geodesic ray $R \subset M$ and the scalar curvature is uniformly positive outside an $r$-neighborhood of $R$ for some $r > 0$, then already

$$\text{ind}_c(D) = 0 \in K_*(C^*(M)).$$
Improved vanishing

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For the proof, consider the small ideal $C^*(R \subset M)$ in $D^*(M)$ of operators in $C^*(M)$ supported near $R$.

- local analysis shows that $\chi(D)$ is invertible module $C^*(R \subset M)$
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- Naturality:

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*If* $M$ *contains a geodesic ray* $R \subset M$ *and the scalar curvature is uniformly positive outside an* $r$-*neighborhood of* $R$ *for some* $r > 0$, *then already*

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- naturality:
  $$K_1(D^*(M)/C^*(M)) \longrightarrow K_0(D^*(M))$$
  $$K_*([0, \infty)) = 0.$$
Theorem (Hanke-Schick)

Let $M$ be a compact spin manifold, $N \subset M$ a submanifold of codimension 2 with trivial tubular neighborhood $N \times D^2 \subset M$.
Assume that every map from a 2-sphere to $M$ can be extended to a map from $D^3$ to $M$ (i.e. $\pi_2(M) = 0$).
Assume that the index of the image of $\pi_1(N)$ in $\pi_1(M)$ is infinite.
Assume that $\text{ind}(D_N) \neq 0$
Then $M$ does not carry a metric of positive scalar curvature.
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Then $M$ does not carry a metric of positive scalar curvature.

Uses the previous result by passing to a suitable covering, then doing a glueing construction along the hypersurface, then uses the partitioned manifold index theorem,...
Throughout, we can replace the complex numbers by any $C^*$-algebra $A$; the algebras $K, B, C^*(M), D^*(M)$ have then (essentially) to be tensored with $A$: we get the K-theory of $A$ into the picture.

The whole story then relates to the *Baum-Connes conjecture*. 

**Example (Gromov-Lawson with different method)**

This applies to the "generic" 3-manifold, namely every orientable prime and irreducible 3-manifold $M$ with submanifold $N$ a non-contractible circle.
Throughout, we can replace the complex numbers by any $C^*$-algebra $A$; the algebras $K, B, C^*(M), D^*(M)$ have then (essentially) to be tensored with $A$: we get the $K$-theory of $A$ into the picture.

The whole story then relates to the *Baum-Connes conjecture*. In the previous theorem, we can replace

\[
\text{ind}(D_N) \neq 0 \text{ by } \text{ind}_{BC}(D) \neq 0 \in K_*(C^*\pi_1(N)).
\]

**Example (Gromov-Lawson with different method)**

This applies to the “generic” 3-manifold, namely every orientable prime and irreducible 3-manifold $M$ with submanifold $N$ a non-contractible circle.
THANK YOU

Or do you want to see technical details of the index construction for the Dirac operator
Index is about kernel/spectrum near zero, so we can push the large "eigenvalues" in to make $D$ bounded: pick

$$\chi(x) = x/(1 + x^2)$$

or (using homotopy invariance of index) any other odd function which goes to $\pm 1$ for $x \to \pm \infty$ and consider $\chi(D)$. By Fourier inversion

$$\chi(D) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\chi}(\xi) \exp(i\xi D) \, d\xi.$$  

Then $\chi(D) \in D^*(M)$. Moreover, $\chi(D)^2 - 1 \in C^*(M)$, i.e. $[\chi(D)^2] = 1 \in D^*(M)/C^*(M)$: invertibility.

If an interval around zero is not in the spectrum, we can choose $\chi$ to have the values $\pm 1$ on the spectrum of $D$, then indeed $\chi^2(D) = 1 \in D^*(M)$ (no quotient necessary).
Proof of analytic properties of $D$

Standard analysis gives

- unit propagation of the wave operator implies that $\exp(i\xi D)$ has propagation $\xi$.
- $(\hat{\chi}^2 - 1)$ and also $\hat{\chi}$ can be approximated by functions with compact support.
- Consequence: finite propagation of $\chi(D)$.
- Elliptic regularity implies the local compactness of $\chi^2(D) - 1$.
- Because $\hat{\chi}(\xi)$ is a distribution which is smooth outside 0, but has a singularity at 0, for $\chi(D)$ in this argument one has to avoid the diagonal and therefore only gets pseudolocality.