On Baum Connes conjecture

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Basic generalisation of a locally compact Hausdorff space is a C*-algebra. The idea is to look at the functor

$$X \rightsquigarrow C_0(X)$$

and replace $C_0(X)$ by a non-commutative C*-algebra.
A C*-algebra is a norm closed subalgebra of $B(H)$ (bounded operators on a Hilbert space H) closed under taking adjoints $a \to a^*$. First examples

1. $M_n(\mathbb{C})$; $H = \mathbb{C}^n$,
2. $C_0(X)$; $h = L^2(X, \mu)$ where μ is any positive Radon measure nonvanishing on any open subset of X and $f \in C_0(X)$ acts by multiplication

$$L^2(X) \ni \xi \to f\xi \in L^2(X).$$

   In fact, any abelian C*-algebra is of this form.
3. $K(H)$ the algebra of all compact operators on H.

The basic norm identity is

$$||a^*a|| = ||a||^2.$$

C*-algebras form a category, with

$$Mor_{C^*}(A, B) = \{\phi : A \to B \mid \phi \text{ is a *-homomorphism}\}.$$

The basic C*-identity implies a sensible notion of positivity, and in particular, every *-homomorphism is automatically continuous. What distinguishes a C*-algebra from complex numbers is the fact that the unit ball is not round.

We can always add some extra structure, f. ex. a G- action

$$\alpha : G \to Aut(A)$$

by *-automorphisms, where G is a (second countable) locally compact group and α is a pointwise continuous homomorphism. In this case

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Topology

The category of Abelian G-C*-algebras coincides with the category of pointed compact Hausdorff G-spaces.

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Example

K-theory is a non-commutative homology theory for C^*-algebras. It maps separable C^*-algebras to the category $\mathfrak{Ab}_c^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded countable Abelian groups.

Example

KK^G is a (bivariant) non-commutative homology theory for C^*-algebras with a G-action.

Cycles in $KK^G(A, B)$

- $\mathcal{H}_B$ is a right Hilbert B-module;
- $\varphi\colon A \to \mathrm{B}(\mathcal{H}_\mathrm{B})$ is a *-representation;
- $F \in \mathrm{B}(\mathcal{H}_\mathrm{B})$;
- $\varphi(a)(F^2 - 1)$, $\varphi(a)(F - F^*)$, and $[\varphi(a), F]$ are compact for all $a \in A$;
- in the even case, γ is a $\mathbb{Z}/2$-grading on $\mathcal{H}_B$;
- $\mathcal{H}_B$ carries a representation U of G which implements action of G and commutes with F up to compacts.

A cycle is trivial, if all the "compacts" above vanish, and two cycles are equivalent, if they are homotopic after adding trivial cycles.

Some properties of KK^G

1. The classes in $\mathrm{KK}_1^G(A, B)$ are given by semisplit extensions: $0 \to B \otimes \mathrm{K} \to E \to A \to 0$
2. Kasparov product $\mathrm{KK}_i^G(A, B) \times \mathrm{KK}_j^G(B, C) \to \mathrm{KK}_{i+j}^G(A, C)$
3. Excision. Given a semisplit short exact sequence $0 \to I \to A \to A/I \to 0$, there exists an associated six term exact sequence

$$\begin{array}{ccccc} \mathrm{KK}_0^G(A/I, B) & \longrightarrow & \mathrm{KK}_0^G(A, B) & \longrightarrow & KK_0^G(I, B) \\ \uparrow & & & & \downarrow \\ KK_1^G(A, B) & \longleftarrow & KK_1^G(A, B) & \longleftarrow & KK_1^G(A/I, B) \end{array}$$

   and similarly in the second variable.
4. For G compact group
   - $KK_*^G(\mathbb{C}, A) = K_*^G(A) = K_*(A \rtimes G)$ - equivariant K-theory
   - $KK_*^G(\mathbb{C}, \mathbb{C}) = R_G$ - the representation ring of G.

Suppose that $G = \mathbb{Z}$. Then
The cycles are given as follows

- An even representation of $\mathbb{Z}$ on a Hilbert space $H = H^+ \oplus H^-$ (hence a pair of unitary operators $U^+ \oplus U^-$),
- A Fredholm operator $F : H^+ \to H^-$ which intertwines U^+ with U^- modulo compacts.

Then the class of (U, F) gives

$$\mathrm{Index(F)} = \dim \ker \mathrm{F} - \dim \operatorname{coker} \mathrm{F} \in \mathbb{Z}.$$

Theorem (BC for $\mathbb{Z}$)

$$KK_0^{\mathbb{Z}}(\mathbb{C}, \mathbb{C}) \ni F \to \mathrm{Index(F)} \in \mathbb{Z}$$

is an isomorphism.

The Kasparov product

$$\mathrm{KK}^G_*(\mathbb{C}, B) \times \mathrm{KK}^G_1(B, C) \to \mathrm{KK}^G_{*+1}(\mathbb{C}, C)$$

has an explicit description as follows.

Given class $[D] \in \mathrm{KK}^G_1(B, C)$, represent it by a semisplit extension

$$0 \to C \otimes \mathrm{K} \to E \to B \to 0.$$

Then the pairing

$$\cap[D] : K^G_*(B) \to K^G_{*+1}(C)$$

coincides with the boundary map δ in the six-term exact sequence

$$\begin{array}{ccccc} K^G_0(C) & \longrightarrow & K^G_0(E) & \longrightarrow & K^G_0(B) \\ \uparrow \delta & & & & \downarrow \delta \\ K^G_1(B) & \longleftarrow & KK^G_1(E) & \longleftarrow & K^G_1(C) \end{array}$$

The universality of Kasparov theory

Theorem (Joachim Cuntz and Nigel Higson)

Bivariant KK*-theory is the universal C^*-stable, split-exact functor on the category of separable C^*-algebras.*
That is, a functor from the category of separable C^*-algebras to some additive category factors through KK *if and only if it is C^*-stable and split-exact, and this factorisation is unique if it exists.*

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Corollary

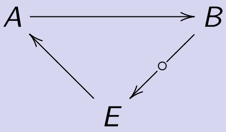
C^*-stability and split-exactness
$\Longrightarrow$ *homotopy invariance, Bott periodicity, Connes–Thom Isomorphism, . . .*

Let KK^G be the category of G-C*-algebras (separable) with morphisms given by KK_0^G (the composition of morphisms is given by Kasparov product.

Theorem

The following gives KK^G triangulated structure

1. *Shift* $A \to SA = C_0(\mathbb{R}, A)$
2. *Exact triangles*



are given by semisplit extensions

$$0 \to SB \to E \to A \to 0$$

Given $A \in KK^G$, look at $A[G]$.

Definition

Set $\alpha : A \to C(G, A)$ to be the *-homomorphism $\alpha(a)(g) = g^{-1}(a)$ The reduced crossed product,

$$A \rtimes_{red} G$$

is the C*-algebra on $A \otimes L^2(G)$ generated by (products of elements in) $\alpha(A)$ and the regular representation of G.

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Definition

The full crossed product, $A \rtimes G$, is the universal enveloping C*-algebra of $A[G]$.

Basic object of study is the functor

$$KK^G \ni A \Rightarrow F(A) = K_*(A \rtimes_{red} G) \in \mathfrak{Ab}^{\mathbb{Z}/2\mathbb{Z}}.$$

This is essentially the functor which describes harmonic analysis for group actions. It is homotopy invariant, but not excisive. Basic reason is the fact the functor $A \Rightarrow A \rtimes_{red} G$ is in general not exact.

"Assembly"

Given a homotopy functor F, construct a homology (excisive) functor $\mathbb{L}F$ and natural transformation $\mathbb{L}F \Rightarrow F$, universal for this situation

We will use the triangulated structure of KK^G.

Let $\mathfrak{I}$ be an ideal in KK^G given by

$$\{j \mid j = 0 \text{ in } KK^H, \text{ for every compact subgroup } H \subset G\}$$

There is the corresponding projective class $\mathcal{P}$ in KK^G, consisting of the collection of algebras P satisfying

$$\mathfrak{I}(A, B) \circ KK^G(P, A) = 0$$

for all A, B.

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Example

1. $\mathfrak{T} = KK^\Gamma$ for a discrete group Γ
2. $j \in \mathfrak{I}$ if, for all torsion subgroups $H \subset \Gamma$, $j = 0$ in KK^H
3. $\mathcal{P}$ coincides with the usual class of proper Γ-algebras.

Theorem

There are enough projectives in KK^G, and, given any $A \in KK^G$, there exists a projective cover

$$P_A \in \mathcal{P},\ D_A \in KK^G(P_A, A)$$

universal for morphisms from $\mathcal{P}$ to A

Definition

K-homology Let $\underline{E}_G$ be the universal proper action of G (it exists!)

$$K^*_G(A) = \lim\{KK^G_*(C(X), A) \mid X \subset \underline{E}_G, X/G \text{ compact}\}$$

In the case when $A = C(M)$ is abelian, this is the usual equivariant K-homology of M.

Theorem

$K_*(P_A \rtimes G) = K^*_G(A)$ *and the assembly for* F *is given by*

$$K^*_G(A) = K_*(P_A \rtimes G) \xrightarrow{D_A} K_*(A \rtimes_{red} G)$$

Baum Connes conjecture

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Status

1. True for discrete groups acting properly isometrically on Hilbert spaces
2. True for almost connected groups (Connes Kasparov conjecture)
3. True for Sp(n,1)
4. Open for SL(3,$\mathbb{Z}$)
5. False for "non-exact groups" (if they exist).

Corollaries of BC

1. Injectivity of assembly implies Novikov conjecture (Higher L-genera are homotopy invariant)
2. Surjectivity of assembly implies Kaplansky conjecture (for torsion free G, $C^*_{red}(G)$ has no nontrivial idempotents.

In general, it is enough to find the "Dirac" element $D = D_{\mathbb{C}}$, since

$$P_A = P_{\mathbb{C}} \rtimes G$$

Remark

Since $P_{\mathbb{C}}$ is a projective cover, there exists an Adams type spectral sequence computing $K_G^*(A)$

G has a γ-element, if $D_{\mathbb{C}} \in KK^G(P_{\mathbb{C}}, \mathbb{C})$ has a left inverse Q, and then $\gamma_G = QD_{\mathbb{C}} \in KK^G(\mathbb{C}, \mathbb{C})$.

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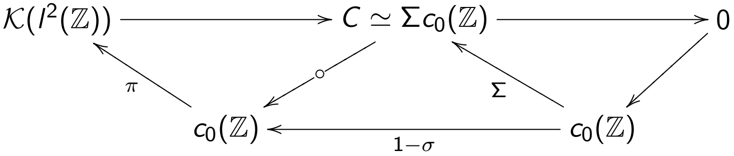
All proofs of BC go via showing that γ_G acts as identity on $K_*(\cdot \rtimes_{red} G)$

G satisfies the strong Baum-Connes conjecture, if $\gamma_G = 1$. This is equivalent to saying that every object in KK^G is in the localizing category generated by the subcategory of projectives.

- $\Gamma = \mathbb{Z}$
- $\mathfrak{J} = Ker: \ KK^{\mathbb{Z}} \to \mathrm{KK}$

The $\mathfrak{J}$-projective resolution of $\mathbb{C}$ has the form



The projective cover of $\mathbb{C} \simeq_{KK^{\mathbb{Z}}} \mathcal{K}(l^2(\mathbb{Z}))$ is just the mapping cone

$$c_0(\mathbb{Z}) \to c_0(\mathbb{Z}) \to \Sigma C_{1-\sigma}.$$

But this is just the rotated exact triangle associated to the extension

$$0 \to \Sigma c_0(\mathbb{Z}) \to C_0(\mathbb{R}) \to c_0(\mathbb{Z}) \to 0,$$

the $*$-homomorphism $C_0(\mathbb{R}) \to c_0(\mathbb{Z})$ given by the evaluation $f \to f|_{\mathbb{Z}}$.

Conclusion

$P_{\mathbb{C}} = C_0(\mathbb{R}^2)$, $D = \overline{\partial}$, the usual Dirac operator (or rather its phase),

$$K^*_{\mathbb{Z}}(A) = K_*((A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z}) \to K_*(A \rtimes \mathbb{Z}),$$

where the assembly map is given by the product with Dirac operator.
The spectral sequence computing $K^*_{\mathbb{Z}}(A)$ becomes the six term exact sequence in K-theory associated to the extension

$$\Sigma(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \rightarrowtail (A \otimes C_0(\mathbb{R}^2)) \rtimes \mathbb{Z} \twoheadrightarrow (A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z}$$

Since $(A \otimes c_0(\mathbb{Z})) \rtimes \mathbb{Z} \simeq A \otimes \mathcal{K}$, this is just the usual Pimsner-Voiculescu exact sequence.