Algebraic index theorems

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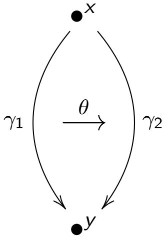
2-groupoids

Data:

1. Units G_0 $\bullet^x$
2. Arrows G_1 $\bullet^x \xrightarrow{\gamma} \bullet^y$
   composable when range of one coincides with the source of the next one.
3. Two-morphisms G_2



with "natural" composition structure (vertical and horisontal).

Differential graded Lie algebras (DGLA)

A DGLA $(L, d, [\ ,\ ])$ is given by the following structure:

- a $\mathbb{Z}$-graded vector space L,
- a differential $d : L_i \to L_{i+1}$ satisfying $d^2 = 0$,
- a bracket $[-,-] : L_i \times L_j \to L_{i+j}$

These satisfy the following:

1. (graded skewsymmetry) $[a, b] = -(-1)^{deg(a)deg(b)}[b, a]$.
2. (graded Jacobi) $[a, [b, c]] = [[a, b], c] + (-1)^{deg(a)deg(b)}[b, [a, c]]$.
3. (graded Leibniz) $d[a, b] = [da, b] + (-1)^{deg(a)}[a, db]$.

2-groupoid of a DGLA

Suppose that $\mathfrak{g}$ is a nilpotent DGLA such that $\mathfrak{g}^i = 0$ for $i < -1$.

A Maurer-Cartan element of $\mathfrak{g}$ is an element $\gamma \in \mathfrak{g}^1$ satisfying

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0. \tag{1}$$

$\mathrm{MC}^2(\mathfrak{g})_0$ is the set of Maurer-Cartan elements of $\mathfrak{g}$.

Think of $\mathrm{MC}^2(\mathfrak{g})_0$ as the set of flat connections

$$d + ad\gamma$$

The unipotent group $\exp \mathfrak{g}^0$ acts on the set of Maurer-Cartan elements of $\mathfrak{g}$ by the gauge equivalences.

arrows

$\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ is the set of gauge equivalences between γ_1, γ_2, with action

$$d + \operatorname{ad} \gamma_2 = \operatorname{Ad} \exp X \,(d + \operatorname{ad} \gamma_1).$$

The composition is given by the product in the group $\exp \mathfrak{g}^0$.

Given $\gamma \in \mathrm{MC}^2(\mathfrak{g})_0$

$$[a,\, b]_\gamma = [a,\, db + [\gamma,\, b]].$$

is a Lie bracket $[\cdot,\cdot]_\gamma$ on $\mathfrak{g}^{-1}$. With this bracket $\mathfrak{g}^{-1}$ becomes a nilpotent Lie algebra. We denote by $\exp_\gamma \mathfrak{g}^{-1}$ the corresponding unipotent group, and by $\exp_\gamma$ the corresponding exponential map $\mathfrak{g}^{-1} \to \exp_\gamma \mathfrak{g}^{-1}$.

2-morphisms

$\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$ is given by $\exp_\gamma \mathfrak{g}^{-1}$ with action

$$(\exp_\gamma t)\cdot(\exp X) = \exp(dt + [\gamma, t])\exp X$$

To summarize, the data described above forms a 2-groupoid which we denote by $\mathrm{MC}^2(\mathfrak{g})$ as follows:

1. the set of objects is $\mathrm{MC}^2(\mathfrak{g})_0$ - Maurer-Cartan elements, or flat connections $d + \gamma$
2. 1-morphisms $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$, are given by the gauge transformations between $d + \gamma_1$ and $d + \gamma_2$.
3. 2-morphisms between $\exp X$, $\exp Y \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ are given by $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$.

A morphism of nilpotent DGLA $\phi : \mathfrak{g} \to \mathfrak{h}$ induces a functor $\phi : \mathrm{MC}^2(\mathfrak{g}) \to \mathrm{MC}^2(\mathfrak{g})$.
However, there are relatively few morphisms of DGLA's. But, since we have to our disposal a differential, we can weaken our conditions, so that they hold up to homotopy.

L_∞ algebras

Let V be a graded vectors space. and denote by CV the (cofree cocommutative coalgebra)

$$\oplus_n S^n(V[1]) = \oplus_n(\Lambda^n V)[n].$$

The coalgebra structure is the one induced from the tensor algebra:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_k (v_1 \otimes \ldots \otimes v_k) \otimes (v_{k+1} \otimes \ldots \otimes v_n)$$

Definition

An L_∞-structure on a graded vector space V is a codifferential Q of degree $+1$ on the graded coalgebra C(V).

Such a Q is just a collection of linear maps

$$Q_n : S^n(V[1]) \to V[1], n \geq 1,$$

of degree 1 such that the coderivation $Q : S(V[1]) \to S(V[1])$ induced by the q_n's by imposing coLeibniz rule is a codifferential, i.e. $Q^2 = 0$.

Just to get familiarized with this notion, let us start with the case of only two operations:

$$q_1 : V[1] \to V[1]; \;\; q_2 : S^2 V[1] \to V[1]; \;\; q_n = 0 \text{ for n>2}.$$

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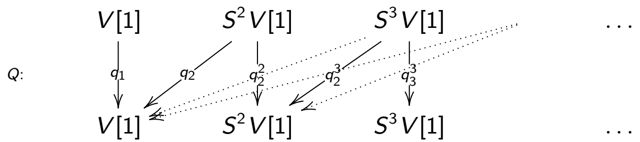

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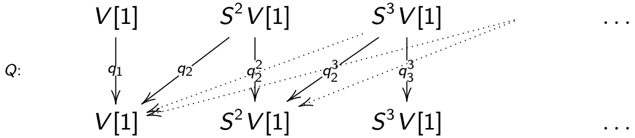

The coderivation property computes q_2^2, q_3^3, q_2^3 e.t.c. in terms of q_1, q_2, For example

$$q_2^2(x\otimes y) = \Delta Q(x\otimes y) = (Q\otimes id + id\otimes Q)\Delta(x\otimes y) = q_1(x)\otimes y \pm x\otimes q_1(y),$$

the dotted arrows are zero and so on.

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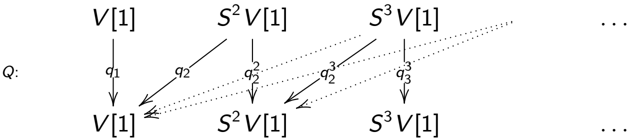

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the dotted arrows are zero and so on.
The equation $Q^2 = 0$ translates into

1. $q_1^2 = 0$
2. $q_1 q_2 = q_2(q_1 \otimes id) + q_2(id \otimes q_1)$
3. $q_2 q_2^3 = 0$

A straightforward check of signs gives

Quillen

Let (V, Q) be an L_∞ algebra with $Q = q_1 + q_2$ (no higher components). Set

- $d = -q_1$,
- $[x, y] = q_2(x \otimes y)$.

Then $(V, d, [\ ,\ ])$ is a DGLA.

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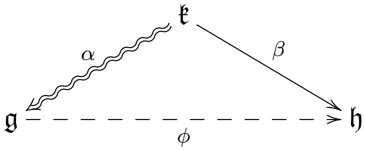
For completeness, any L_∞-morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ is part of a commutative triangle



where $\mathfrak{k}$ is a DGLA, α and β are DGLA morphisms and α is a quasiisomorphism.

The following holds:

Theorem

Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are DGLA's, and that $\phi : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of L_∞ algebra structures. Let $\mathfrak{m}$ be a nilpotent commutative ring. Then the induced map $\phi : \mathrm{MC}^2(\mathfrak{g} \otimes \mathfrak{m}) \to \mathrm{MC}^2(\mathfrak{h} \otimes \mathfrak{m})$ is an equivalence of 2-groupoids.

Examples

Suppose that A is a k-algebra with associative product m. The k-vector space $C^n(A)$ of Hochschild cochains of degree $n \geq 0$ is defined by

$$C^n(A) := \mathrm{Hom}_k(A^{\otimes n}, A) \ .$$

The graded vector space $C^*(A)[1]$ has a canonical structure of a DGLA under the Gerstenhaber bracket denoted by $[\ ,\ ]$ and differential $\delta = adm$.

$C^*(A)[1]$ is canonically isomorphic to the (graded) Lie algebra of derivations of the free associative co-algebra generated by $A[1]$.

Let R be a commutative Artin k-algebra with maximal ideal $\mathfrak{m}_R$. There is a canonical isomorphism $R/\mathfrak{m}_R \cong k$.

A *(R-)star product* on A is an associative R-bilinear product on $A \otimes_k R$ such that the canonical isomorphism of k-vector spaces $(A \otimes_k R) \otimes_R k \cong A$ is an isomorphism of algebras. Thus, a star product is an R-deformation of A.

Deformation functor Def(A)

The 2-category of R-star products on A, denoted Def$(A)(R)$, is the 2-groupoid given as follows:

- Objects: m,
  R-star products on A,
- 1-morphisms $\phi : m_1 \to m_2$ R-algebra homomorphisms $\phi : (A \otimes_k R, m_1) \to (A \otimes_k R, m_2)$ which reduce to the identity map modulo $\mathfrak{m}_R$.
- 2-morphisms $b : \phi \to \psi$.
  Elements $b \in 1 + A \otimes_k \mathfrak{m}_R \subset A \otimes_k R$ such that $m_2(\phi(a), b) = m_2(b, \psi(a))$ for all $a \in A \otimes_k R$.

It is clear that the assignment $R \mapsto$ Def$(A)(R)$ extends to a functor on the category of commutative Artin k-algebras.

Suppose that μ is an R-star product on A. Since

$$\omega = \mu - m = 0 \mod \mathfrak{m}_R,$$

$\omega \in C^2(A) \otimes_k \mathfrak{m}_R$.

The associativity of μ is equivalent to the fact that ω satisfies the Maurer-Cartan equation, i.e.
$\mu - m \in \mathrm{MC}^2(C^*(A)[1] \otimes_k \mathfrak{m}_R)_0$.

Algebraic index theorems

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It is easy to see that the assignment $\mu \mapsto \mu - m$ extends to a functor

$$\mathrm{Def}(A)(R) \to \mathrm{MC}^2(C^*(A)[1] \otimes_k \mathfrak{m}_R)\ . \qquad (2)$$

The following proposition is obvious.

Theorem

The functor (2) is an isomorphism of 2-groupoids.

In particular, we get a bijection

$$\left\{ \begin{array}{c} \text{R-star products on A} \\ \text{modulo isomorphisms} \end{array} \right\}$$

$$\Updownarrow$$

$$\left\{ \begin{array}{c} \text{Maurer-Cartan elements of } C^*(A)[1] \otimes_k \mathfrak{m}_R \\ \text{modulo gauge equivalence} \end{array} \right\}$$

Variants of $C^*(A)[1]$

1. Suppose that M is a smooth manifold, and $A = C^\infty(M)$. The subDGLA $C^*_{diff}(A)[1]$ of continuous (in the C^∞ topology) cochains controls $*$-deformations of M, i.e. associative products on $C^\infty(M)[[\hbar]]$ satisfying

$$f * g = fg + \hbar P_1(f,g) + \hbar^2 P_2(f,g) + \ldots,$$

where P_k are bidifferential operators.

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2. Suppose that M is complex analytic. Denote by $\mathcal{O}_M$ the structure sheaf of M (the sheaf of holomorphic functions). Given open subset U of M, denote by $C^*_{hol}[1](U)$ the DGLA of Hochschild cochains on $\Gamma(U, \mathcal{O}_U)$ given by holomorphic polydifferential operators on U. The sheaf of DGLA's

$$U \to C^*_{hol}[1](U)$$

controls the deformations $*$-deformations of M. Note that these consist of simultaneously deforming the structure sheaf of M and deforming the local product of functions.

Polyvectorfields

Suppose that M is a (say compact) smooth manifold Then

$$\Gamma(M, \Lambda^* TM)[1]$$

is a DGLA with trivial differential, and Schouten bracket. The associated Deligne 2-groupoid of it consists of

1. the Maurer Cartan elaments are $\pi \in \Gamma(M, \Lambda^2 TM)$ satisfying
$$[\pi, \pi] = 0$$
i. e. Poisson structures.
2. the 1-morphisms are just vector fields, and the action of X on π is just by the diffeomorphism $exp(X)$
3. the 2-morphisms coincide with $C^\infty(M)$, with the bracket $[f, g]_\pi = \pi(f, g)$, and they act on 1-morphisms by
$$X \to log(exp(H_f expX),$$
where H_f is the Hamiltonian vector field $H_f = \iota_{df}\pi$ and the "log" refers to Hausdorff-Campbell formula.

Note that the Hochschild cohomology of $H^*(C^\infty(M), C^\infty(M))[1]$ is a DGLie algebra, and coincides with $\Gamma(M, \Lambda^* TM)[1]$.. The following is he celebrated formality theorem.

Theorem (Kontsevich)

There exists an L_∞ quasiisomorphism

$$\Gamma(M, \Lambda^* TM) \to C^*_{diff}(C^\infty(M))[1].$$

In particular, $*$-deformations of M are in bijection with equivalence classes of formal Poisson structures on M.

The same result holds in much greater generality.

1. M is a complex manifold. Then there exists a L_∞-quasiisomorphism of DGLA's

$$\bigoplus_* \Omega^{0,\bullet-*}(M, \Gamma_{hol}(\Lambda^* T_{hol} M))[1] \to C^\bullet_{hol}(\mathcal{O}_M)[1]$$

The Maurer-Cartan elements on the left hand side are

$$\pi \in \bigoplus_* \hbar\Omega^{0,2-*}(M, \Gamma_{hol}(\Lambda^* T_{hol} M))$$

satisfying

$$\overline{\partial}\pi + \frac{1}{2}[\pi, \pi] = 0$$

and these classify the deformations of $\mathcal{O}_M$ as a sheaf of categories.

2. M is a smooth (or complex) manifold, and $c \in H^3(M, \mathbb{Z})$ defines a gerbe on M.
   1. The deformation complex of (M, c) is, roughly, the same as the deformation complex of the bundle of infinite matrices $Mat^\sigma(C^\infty(M))$ over M twisted by the class c. $C^*(Mat^\sigma(C^\infty(M)), Mat^\sigma(C^\infty(M)))[1]$ is a sheaf of DGLA's.
   2. Let ω be a differential form representing the class $[c]$. Then we get a L_∞ algebra

$$(\Gamma(M, \Lambda^* TM)[1], [\ ,\ ], l_\omega)$$

where $[\ ,\ ]$ is, as before, the Schouen bracket, and l_ω is a ternary operation

$$\Lambda^{n_1} TM \wedge \Lambda^{n_2} TM \wedge \Lambda^{n_3} TM \to \Lambda^{n_1+n_2+n_3-3} TM$$

defined by contracting ω with a single vector from each of the polyvectorfields $\Lambda^{n_1} TM$, $\Lambda^{n_2} TM$, $\Lambda^{n_3} TM$.

Theorem (Formality theorem for gerbes)

There exists an L_∞-quasiisomorphism

$$C^*_{diff}(Mat^\sigma(C^\infty(M)), Mat^\sigma(C^\infty(M)))[1] \to (\Gamma(M, \Lambda^* TM)[1], [\ ,\ ], I_\omega)$$

The Maurer-Cartan elements of the DGLA on the right hand side exist precisely when ω is exact, and are given by 2-vectorfields $\pi \in \hbar\Lambda^2 TM[[\hbar]]$ satisfying

$$[\pi, \pi] = I_\omega(\pi \wedge \pi \wedge \pi).$$

Hence

Stack deformations of a gerbe c on a (smooth or analytic) manifold M exist iff $[c]$ is torsion, and are in bijective correspondence with equivalence classes of twisted Poisson structures.

Index problem

Once we have a deformed algebra, say $(\mathcal{A}_M^\hbar, *)$ as a deformation of $C^\infty(M)$ over $k = \mathbb{C}[[\hbar]]$, the next problem is to compute it's K-theory. Actually, we are interested in a more precise question:

Algebraic index problem

Given a trace (or cyclic periodic cocycle) τ on $\mathcal{A}_M^\hbar(C^\infty(M)[[\hbar]], *)$, compute the pairing

$$< \tau, \bullet >: K_0(\mathcal{A}_M^\hbar) \to k.$$

The procedure is usually to go via Chern character

$$Ch : K_*(\mathcal{A}_M^\hbar) \to CC_*^-(\mathcal{A}_M^\hbar) \to CC_*^{per}(\mathcal{A}_M^\hbar)$$

and then compute explicitly the pairing on cyclic homology and cohomology.

Recall the Goodwillie's theorem, which says that cyclic periodic (co)homology is actually independent of the deformation (both are invariant under nilpotent extensions), hence there is an easy "reduction to $\hbar = 0$" quasiisomorphism of complexes

$$CC_*^{per}(\mathcal{A}_M^\hbar) \xrightarrow{\textit{principal symbol}} CC_*^{per}(\mathcal{O}_M[[\hbar]])$$

$$\downarrow \textit{Hochschild Kostant Rosenberg}$$

$$(\Omega^*(M)[[u, u^{-1}], ud),$$

maybe with some variations on the (periodicized) de Rham complex in the more general cases like that of complex manifold or gerbe.

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Hence the real question is to identify concrete cyclic cocycles on $\mathcal{A}_M^\hbar$ as currents on M.

Examples

1. (M, ω) is a symplectic manifold. Let ϖ be the associated Poisson structure, and we are interested in deformations along ϖ, i. e. such that $f * g = fg + \frac{i\hbar}{2}\varpi(df, dg) + O(\hbar^2)$. The motivations for this example are for example the pseudodifferential calculus on a smooth manifold, where w are aiming at the (analogue of) Atiyah-Singer theorem, or representation theory of compact Lie groups,where we are aiming at the Weyl character formula.
2. $M = T^*X$, where X is a complex manifold. The source is the calculus of microdifferential perators, and the goal is the Riemann-Roch theorem for D-modules. More about it later.

3. M is a smooth manifold with a gerb c. The range of the Chern character, the periodic cyclic homology of $Mat^\sigma(C^\infty(M))$ (deformed or not) is given by the twisted de Rham cohomology

$$(\Omega^*(M)[u^{-1}, u]], u(d + \omega)),$$

where again ω represents the class of $[c]$ in $H^3_{DR}(M)$. In the case when deformations exist, one can choose $\omega = 0$. The result here is the algebraic version of Mathai-Melrose-Singer index theorem for pseudodifferential operators on a gerbe.

The general framework for the algebraic index theorem is the **formality for chains**. We will start with freshman calculus on a smooth manifold. The basic structure involved consists of

1. DGLA $\Gamma(M, \Lambda^* TM)[1]$, and
2. complex $(\Omega^*(M), d)$.

Polyvector fields act on differential fields via Lie derivative $\mathcal{L}$ and contraction ι. The identities

i. $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$, $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, $[\iota_X, \iota_Y] = 0$, and
ii. $[\iota_X, d] = \mathcal{L}_X$, $[\mathcal{L}_X, d] = 0$

can be translated as the following. Set
$|\epsilon| = 1$, $\epsilon^2 = 0$, $|u| = -2$,

$$\mathcal{L}^* = ((\Gamma(M, \Lambda^* TM)[1][u, \epsilon], [\ ,\ ], u\frac{\partial}{\partial \epsilon});\ C_{-*} = (\Omega^*[[u]], ud)$$

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Then C_{-*} is a module over the DGLA $\mathcal{L}^*$.

The analogue of polyvectorfields is, as we have seen, the shifted Hochschild cohomological complex and the analogue of de Rham complex is the cyclic periodic complex

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Basic fact of life

Given an algebra A over field of characteristic 0, the cyclic periodic complex

$$(CC_{-*}(A), b + uB)$$

is an L_∞-module over the DGLA

$$(C^*(A)[1][\epsilon, u], u\frac{\partial}{\partial \epsilon})$$

.

Let's go back to a smooth manifold case.

Theorem (Formality for chains, Tamarkin, Tsygan)

Given "formality" L_∞ quasiisomorphism

$$\phi : \Gamma(M, \Lambda^* TM) \to C^*_{diff}(C^\infty(M))[1]$$

there exists a quasiisomorphism of L_∞-modules

$$\Phi : (CC^-_*(C^\infty(M))[[u]], b + uB) \to (\Omega^{-*}[[u]], ud)$$

compatible with ϕ.

Let us apply this to a Poisson structure π and the associated Maurer-Cartan element Π in $C^2_{diff}(C^\infty(M))$. As we have seen, Π defines a star product $*_\pi$ in $C^\infty(M)$, and we will denote the induced algebra structure (over k=$\mathbb{C}[[\hbar]]$), by $\mathcal{A}_\pi$. By the L_∞-equivariance, the map Φ above intertwines $ud + \hbar L_\pi$ and $b + B + \hbar L_\Pi$. It is immediate to check that the Hochschild boundary map of $\mathcal{A}_\pi$, b_*, is related to the Hochschild boundary map b of non-deformed algebra by

$$b_* = b + \hbar L_\Pi.$$

In other words, Φ becomes a quasiisomorphism

$$\Phi : (CC_{-*}(\mathcal{A}_\pi)[[u]], b_* + uB) \to (\Omega^{-*}[[u]], ud + \hbar L_\pi)$$

But L_π is contractible on the de Rham complex hence, after localizing in u, we get a quasiisomorphism

Local trace density

$$TR : CC^{per}_{-*}(\mathcal{A}_\pi) \to (\Omega^{-*}[[u]]_u, ud)$$

This is in fact the "natural" trace, the one which appears in applications.

On the other hand, recall the other morphism of the cyclic periodic complex to de Rham complex, the one which came via reduction $\hbar = 0$, and which is explicitly computable reduction $\hbar = 0$

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So what is really at stake is the (non)-commutativity of the following diagram:

$$\begin{array}{ccc} CC^{per}_{-*}(\mathcal{A}_\pi) & \xrightarrow{\quad TR \quad} & (\Omega^{-*}[[u]]_u, ud) \\ {\scriptstyle principal\ \ symbol}\downarrow & & \uparrow {\scriptstyle =} \\ CC^{per}_{-*}(\mathcal{A}_0) & \xrightarrow[\quad Hochschild\ \ Kostant\ \ Rosenberg \quad]{} & (\Omega^{-*}[[u]]_u, ud) \end{array}$$

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CC_{-*}^{per}(\mathcal{A}_0) & \xrightarrow[\textit{Hochschild Kostant Rosenberg}]{} & (\Omega^{-*}[[u]]_u, ud)
\end{array}$$

Formal case

The formality maps are not unique, they depend on a choice of an associator. Hence in practice we need a more tight structure to pin down, what is the top map in the above diagram and how to make it commutative.
An example of this is the case of deformations of symplectic structures.

First some notation.

1. $\mathbb{O} = (k[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\xi}_1, \ldots \hat{\xi}_n]], \cdot)$, the algebra of formal power series in $2n$ variables over $\mathbb{C}[[\hbar]]$;
2. $\mathbb{W} = (k[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\xi}_1, \ldots \hat{\xi}_n]], *)$, where $*$ is the Weyl product, satisfying the usual relations:

$$[\hat{x}_i, \hat{x}_j] = 0,\ [\hat{\xi}_i, \hat{\xi}_j] = 0,\ [\hat{\xi}_i, \hat{x}_j] = \frac{i\hbar}{2}\delta_{i,j};$$

3. $\mathbb{O}$ has a structure of a Poisson algebra, with Poisson bracket $\hat{\pi}$ induced by the symplectic structure

$$\hat{\omega} = \sum_i d\hat{x}_i d\hat{\xi}_i$$

Recall the diagram TR.

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In our formal case, it becomes

$$\begin{array}{ccc} CC^{per}_{-*}(\mathbb{W})_\hbar & \xrightarrow{TR} & \hat{\Omega}^*_\hbar \\ \downarrow{\scriptstyle \hbar=0} & & \uparrow{\scriptstyle =} \\ CC^{per}_{-*}(\mathbb{O})_\hbar & \xrightarrow[HKR]{} & \hat{\Omega}^*_\hbar \end{array}$$

The top map TR is THE canonical quasiisomorphism induced by

$$1 \to \frac{1}{(2\hbar)^n} d\hat{\xi}_1 \wedge d\hat{x}_1 \ldots \wedge d\hat{\xi}_n \wedge d\hat{x}_n$$

All three complexes are quasiisomorphic, and the diagram commutes.

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All three complexes are quasiisomorphic, and the diagram commutes. In fact cohomology of all three complexes is $\mathbb{C}[\hbar^{-1}, \hbar]]$.

What will become important very soon is the fact that our complexes have some equivariance.

So let us give a few more definitions.

1. Set $\mathfrak{g} = Der\mathbb{W}$, $\tilde{\mathfrak{g}} = \frac{1}{\hbar}\mathbb{W}$. Both are Lie algebras over $\mathbb{C}$, where the bracket in $\tilde{\mathfrak{g}}$ is defined by $[\frac{1}{\hbar}f, \frac{1}{\hbar}g] = \frac{1}{\hbar^2}(f * g - g * f)$. They fit into the Lie algebra extension

$$\hat{\theta} : 0 \longrightarrow \frac{1}{\hbar}k \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0 ,$$

which defines a class in $H^2(\mathfrak{g}, \mathbb{C})$;

2. $\hat{\Omega}$ a module over $\mathfrak{g} \ltimes \tilde{\mathfrak{g}}[1]$ by sending $F + G[1]$ to $\mathcal{L}_{\hat{H}_{\hbar F}} + \iota_{\hat{H}_{\hbar G}}$, where the (formal) Hamiltonian vector field $\hat{H}_F$ is given by $\hat{\pi}(dF, \cdot)$;

3. $CC_{-*}(\mathbb{W})$ is a $\mathfrak{g} \ltimes \tilde{\mathfrak{g}}[1]$-DGLA module by inclusion

$$\mathfrak{g} \ltimes \tilde{\mathfrak{g}}[1] \hookrightarrow C^*(\mathbb{W}, \mathbb{W})[1];$$

4. $\mathbb{O}$ is a module over $\mathfrak{g} \ltimes \hbar\tilde{\mathfrak{g}}[1]$, where, given $F + \hbar G[1] = F_0 + a[1] \mathrm{mod} \hbar$, we let F_0 act by $\mathcal{L}_{H_{F_0}}$ and $a[1]$ by the shuffle with $1 \otimes a$.

Analytic and topological traces in Lie algebra cohomology

TR extends to a cohomology class $\hat{\tau}_a$ and
the composition $HKR \circ (\hbar = 0)$ extends to a cohomology class $\hat{\tau}_{top}$,
both in the group

$$\mathbb{H}^0_{Lie}(\mathfrak{g} \ltimes \hbar\tilde{\mathfrak{g}}[1], Hom(CC^{per}_{-*}(\mathbb{W})_\hbar, (\hat{\Omega}^*[u^{-1}, u]]_\hbar, ud)));$$

The class $\hat{\tau}_a$ is called the local trace density, for reasons which will be apparent later.

Theorem

The equality

$$\hat{\tau}_a = \hat{\tau}_{top} \sum_p u^p \left(\sqrt{\widehat{A}} e^{\hat{\theta}} \right)_{2p}$$

holds in the cohomology group

$$\mathbb{H}^0_{Lie}(\mathfrak{g} \ltimes \hbar^2 \tilde{\mathfrak{g}}[1], Hom(CC^{per}_{-*}(\mathbb{W})_\hbar, (\hat{\Omega}^*[u^{-1}, u]]_\hbar, ud)));$$

Here $\widehat{A}$ is the $\widehat{A}$ polynomial of the Chern classes in $H^*_{Lie}(\mathfrak{g} \ltimes \tilde{\mathfrak{g}}[1], k)$ induced from the inclusion of U(n) as a maximal compact subgroup of Sp(2n).

Let $\mathbb{W}_M = \mathrm{Sym}\,[[T_M^*]]\,[[\hbar]]$ with the $\mathrm{Sp}\,(T_M)$-equivariant Moyal-Weyl product. Recall that this is the bundle

$$\begin{array}{ccc} \mathbb{W} & \longrightarrow & \mathbb{W}_M \\ & & \downarrow \\ & & M \end{array}$$

with the fiber at the point $m \in M$ given by the Weyl algebra of the symplectic vector space $(T_m^*(M), \omega_m^t)$:

$$\prod_n T_m^*(M)^{\otimes n} \,/\, \{\xi \otimes \eta - \eta \otimes \xi = i\hbar\omega_m^t(\xi, \eta)\}\,.$$

with the symplectic structure ω_m^t on $T_m^*(M)$ induced by the symplectic structure ω_m on $T_m(M)$.

Recall the basic result about Fedosov quantization.

Theorem (Construction)

Let ∇ be a flat, $\mathfrak{g}$-valued connection in $\mathbb{W}_M$ of the form

$$\nabla = \frac{1}{\hbar}[I_\omega, \cdot] + \nabla_0 + O(\hbar),$$

where $I_\omega : TM \to T^*M$ be the isomorphism provided by the symplectic structure on M and ∇_0 is any symplectic connection on TM. Then

$$\mathcal{A}_M = \{f \in \Gamma(M, \mathbb{W}_M) \mid \nabla f = 0\}$$

is bijective with $C^\infty(M)$ and is a deformation quantization of M along ω.

Theorem (Existence and classification)

Moreover, given an lifting $\tilde{\nabla}$ of ∇ as above to a $\tilde{\mathfrak{g}}$-valued connection,

$$\theta = [\tilde{\nabla}] \in \frac{\omega}{i\hbar} + H^2(M, k)$$

and isomorphism classes of formal deformations of $C^\infty(M)$ are in bijection with the classes θ modulo symplectomorphisms of M.

Remark

Similar statement holds both for deformation quantization of complex analytic manifolds and for deformation quantization of gerbes. The case of gerbe is a slight generalization, in fact, instead of a flat connection, we get a pair (∇, R), where $R \in \Omega^2(M, \mathbb{W}_M)$ satisfying

$$\nabla^2 = R \text{ and } \nabla(R) = 0.$$

In this case the class θ is given by $[\tilde{\nabla}^2 - R]$.

So suppose that we do have a deformation quantization $\mathcal{A}_M$ of a *symplectic manifold* M*. By above, it comes with a* $\mathfrak{g}$*-valued flat connection* ∇ *(or more generally, with a pair* (∇, R)*).*
Let $(\mathbb{L}^*, \partial_{\mathbb{L}})$ *be a* $\mathfrak{g} \ltimes \hbar^2\tilde{\mathfrak{g}}\,[1]$*-module. Let*

$$\mathcal{L}^* = \mathcal{P}_{\mathrm{symp}} \times_H \mathbb{L}^*$$

be the associated graded vector bundle on M *(here* $\mathcal{P}_{\mathrm{symp}}$ *is the bundle of symplectic frames). Define the differential on* $\Omega^*(M, \mathcal{L}^*)$ *as follows. In local coordinates, if* $\nabla = d_{\mathrm{DR}} + A$*, the differential is defined as*

$$d_{\mathrm{DR}} + \mathcal{L}_A + \mathcal{L}_{\underline{R}} + \partial_{\mathbb{L}}.$$

Denote by $C^*_{\mathrm{Lie}}\left(\mathfrak{g} \ltimes \hbar^2\tilde{\mathfrak{g}}\,[1]\,, \mathfrak{h}; \mathbb{L}^*\right)$ *the complex of relative Lie cochains, with the differential given by the sum of Lie cohomology coboundary and* $\partial_{\mathbb{L}}$*.*

1. $\mathbb{L}^* = \mathbb{C}$, with trivial action. Then $\Omega^*(M, \mathcal{L}^*)$ is just the de Rham complex of M.
2. suppose that $\mathbb{L}^*$ is the complex $CC_*^{per}(\mathbb{W})_\hbar$. Then the complex $\Omega^*(M, \mathcal{L}^*)$ becomes

$$\left(\Omega^*\left(M, CC_*^{per}\left(\mathbb{W}_M\right)\right), \nabla + b + uB + \mathcal{L}_{\underline{R}}\right)$$

Gelfand-Fuks morphism

The main idea of "formal geometry" is the following observation

$$GF(\varphi) = \varphi(A + \underline{R}, A + \underline{R}, \ldots, A + \underline{R})$$

defines a morphism of complexes

$$GF \colon C^*_{\mathrm{Lie}}\left(\mathfrak{g} \ltimes \hbar^2 \tilde{\mathfrak{g}}[1], \mathfrak{h}; \mathbb{L}^*\right) \to \Omega^*(M, \mathcal{L}^*)$$

We use notations $X = (X, 0)$ and $\underline{a} = (0, a)$ for elements of $\tilde{\mathfrak{g}}$.

Theorem

Let

$$\tau_a = GF(\hat{\tau}_a) \text{ and } \tau_{top} = GF(\hat{\tau}_{top}).$$

both as morphisms of complexes

$$\left(\Omega^*\left(M, CC^{per}_{-*}\left(\mathbb{W}_M\right)\right), \nabla + b + uB + L_R[1]\right)$$
$$\big\downarrow$$
$$\left(\Omega^{-*}(M)[u^{-1}, u]], ud\right)$$

Then

$$\tau_a = \tau_{top} \sum_p u^p \left(\sqrt{\widehat{A}} e^{\hat{\theta}}\right)_{2p}$$

For the proof, we just apply the Gelfand-Fuks map to the Lie algebra index theorem. τ_a is the trace density, as it associates to a cyclic periodic chain a differential form on M.

Just some examples.

1. X smooth compact, $M = T * X$ with its canonical symplectic structure ω. The deformation quantization of M is basically a symbol calculus of pseudodifferential operators on X. To be more explicit, choose some metric on X and set, for a Schwartz function $f \in \mathcal{S}(M)$,

$$Op_\hbar(f) : g \to \int_X \chi(x,y) \int_{(T^*_x X)} f(x, \hbar\xi) e^{i<\xi, exp_x^{-1}(y)>} g(x) d\xi dy$$

where $\chi(x,y)$ is a cutoff to a geodesic neighbourhood of the diagonal and $exp_x : T_x X \to X$ is the exponential map. Then $Op_\hbar(f)$ is smoothing and the asymptotics of

$$Op_\hbar(f) Op_\hbar(g) \simeq Op_\hbar(f * g) \text{ mod } \hbar^\infty$$

provide a star product $f * g$ on M along ω. We will again denote the corresponding deformed algebra of functions by $\mathcal{A}_M$. The asymptotics of the standard trace $Tr(Op_\hbar(f)) = \frac{1}{\hbar^n} \int_M f dx d\xi$ provide a trace τ on $\mathcal{A}_M$.

In fact

Atiyah - Singer

Let $c \in CC^{per}(\mathcal{A}_M)$, and denote by c_0 its reduction modulo $\hbar$. Then

$$< \tau, c >= \int_M \tau_a(c)$$

and the algebraic index theorem reduces to

$$< \tau, c >= \int_M HKR(c_0)\sqrt{\hat{A}_M}$$

In particular, given $e \in K_0(\mathcal{A}_M)$ and if $e_0 \in K^0(M)$ is its reduction modulo $\hbar$, we get

$$< \tau, ch(e) >= \int_M ch(e_0)\sqrt{\hat{A}_M}$$

2 A slight variation on the theme. Let $\phi(x_0, \ldots, x_k)$ be a Alexander-Spanier cohomology class on M, and let $K_\hbar^f)$ be the kernel of $Op_\hbar(f)$. Then the asymptotics at $\hbar = 0$ of

$$f_0 \otimes \ldots \otimes f_k \to \int_M \ldots \int_M K_\hbar^{f_0}(x_0, x_1) K_\hbar^{f_1}(x_1, x_2) \ldots K_\hbar^{f_k}(x_k, x_0) \phi(x_0, \ldots, x_k)$$

define a cyclic cohomology class τ_ϕ on $\mathcal{L}_M$.
Note that, while τ_ϕ is not a cyclic cocycle on smoothing operators, it still has pairing to the image of the chern character, since the associated homology classes are supported arbitrarily close to the diagonal $x_0 = x_1 = \ldots x_k$. This gives

Connes Moscovici index theorem

$$\tau_\phi(f_0 \otimes \ldots \otimes f_k) = \int_M \tau_a(f_0 \otimes \ldots \otimes f_k).$$

The algebraic index theorem says here

$$\tau_\phi(f_0 \otimes \ldots \otimes f_k) = \int_M f_0 df_1 \ldots df_k \tilde{\phi} \sqrt{\hat{A}_M} \bmod \hbar$$

and, for a class $e \in K_0(\mathcal{A}_M)$,

$$< \tau_\phi, e >= \int_M ch(e_0) \tilde{\phi} \sqrt{\hat{A}_M}$$

We denoted by $\tilde{\phi}$ the differential form representing the class of ϕ.

3. Another example is the holomorphic case. Let X be a complex manifold and $M = T^*X$ its cotangent bundle, with its standard, holomorphic symplectic form. since there are very few global holomorphic sections, we have to work with the sheaf version of the deformation quantization. The analogue of pseudodifferential operators in this case are microdifferential operators.

Our basic objects are $\mathcal{D}_X$-modules, where $\mathcal{D}_X$ is the sheaf of holomorphic differential operators on X. $\mathcal{D}$ is filtered by degree. This gives immediately a deformation quantization of M as follows.

Set

$$R_{\mathcal{D}} = \bigcup_k \{(d_0, \ldots, d_k, 0, \ldots) \mid d_i \in \mathcal{D}^i\}.$$

This is a ring (with "convolution product"), and we let $\hbar$ act by sticking extra 0 at the beginning of the sequence. This makes into a flat $\mathbb{C}[[\hbar]]$-module, and

$$R_{\mathcal{D}} \simeq \prod_k \hbar^k R_{\mathcal{D}} / \hbar^{k+1} R_{\mathcal{D}}$$

as sheaves over $\mathbb{C}[[\hbar]]$.

The right hand side is locally isomorphic to the ring of polynomial functions on the cotangent bundle, and any choice of such an isomorphism above produces a $*$-product on the sheaf of holomorphic functions on M (the sheaf structure itself is also deformed). One checks that the characteristic class of this deformation is $\frac{1}{2}c_1(M)$. We will denote this deformed sheaf of algebras by $\mathcal{A}_\hbar$ as before. But one needs to stress that it is a sheaf and not just a ring any more.

Suppose now that $\mathcal{M}$ is good $\mathcal{D}$-module (has nice filtration compatible with the filtration by degree). Then, given such a module,

$$\mathcal{M} \otimes_{\mathcal{D}} R_{\mathcal{D}}$$

makes it a module over the deformed algebra, and

$$\sigma(\mathcal{M}) = \mathcal{M} \otimes Gr(R_{\mathcal{D}})$$

is a module over the undeformed structure sheaf of M (which we will denote by $\mathcal{A}_0$.

Riemann-Roch theorem relates to perfect $\mathcal{D}_X$-modules, and it computes image of $id|_{\mathcal{M}}$, under the composition:

$$
\begin{array}{ccc}
\Gamma(End_{\mathcal{D}}(\mathcal{M})) & \longrightarrow & R\Gamma(End_{\mathcal{D}}(\mathcal{M})) \\
\uparrow & & \uparrow \scriptstyle{iso\ \ for \mathcal{M}\ \ perfect} \\
\Gamma(\mathcal{M} \otimes_{\mathcal{D}} \mathcal{M}^*) & \longrightarrow & R\Gamma(\mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{M}^*) \\
& & \scriptstyle{iso\ \ by\ \ abstract\ \ nonsense} \downarrow \\
& & R\Gamma(\mathcal{M} \otimes_{\mathbb{C}} \mathcal{M}^*) \otimes^{\mathbb{L}}_{\mathcal{D}^e} \mathcal{D} \\
& & \scriptstyle{Denis\ \ trace\ \ map} \downarrow \\
& & R\Gamma(\mathcal{D} \otimes^{\mathbb{L}}_{\mathcal{D}^e} \mathcal{D}) \\
& & \scriptstyle{iso\ \ by\ \ Brylinski} \downarrow \\
& & R\Gamma(\mathbb{C}_{T^*X})
\end{array}
$$

This class is called the "Microeulerclass" $(\mu\chi)$, and its integral computes the Euler class of the de Rham complex of $\mathcal{M}$ (which is finite precisely when $\mathcal{M}$ is perfect).

The bottom of the diagram above is just de Rham cohomology of M and, if we note that Hochschild homology (second complex from the bottom) is the same as cyclic periodic homology, what we are really computing is the trace of identity on $\mathcal{M}$, i.e.

$$\mu\chi(\mathcal{M}) = \tau_a(ch(\mathcal{M}))$$

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$$\mu\chi(\mathcal{M}) = \tau_a(ch(\mathcal{M}))$$

By the algebraic index theorem, we get

Theorem (Riemann-Roch for $\mathcal{D}$-modules)

Given a perfect $\mathcal{D}_X$ module $\mathcal{M}$, it's microeuler class is given by

$$\mu\chi(\mathcal{M}) = ch(\sigma(\mathcal{M}))Td_X$$