SYM Lecture 27/6/2012

Failure of excision in algebraic K-theory and what to do about it ...

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A : unital associative ring

$\mathcal{P}(A)$: category of (small) f.g. proj. right A-modules

Grothendieck group :

$$K_0(A) = \frac{\text{free abelian grp. gen. by } \operatorname{ob} \mathcal{P}(A)}{\begin{array}{l}\text{subgrp. w. one gen. } P' - P + P'' \\ \text{for every short exact sequence} \\ 0 \to P' \to P \to P'' \to 0 \quad \text{in } \mathcal{P}(A)\end{array}}$$

Ex 1) k field, $K_0(k) \xrightarrow[\sim]{rk} \mathbb{Z}$.

2) $[K : \mathbb{Q}] < \infty$ number field,

$$0 \to \operatorname{Pic}(\mathcal{O}_K) \to K_0(\mathcal{O}_K) \underset{\longleftarrow}{\xrightarrow{rk}} \mathbb{Z} \to 0$$ //

Algebraic K-theory (symmetric) spectrum

$$K(A) = \{ K(A)_n ,\ K(A)_m \wedge S^n \to K(A)_{m+n} \}.$$

Quillen higher algebraic K-groups

$$K_q(A) = [S^q , K(A)]$$

$\uparrow$

morphisms in homotopy category of symm. spectra

Ex 1) If $[K:\mathbb{Q}] < \infty$ totally real, then

$$\zeta_K(1-2k) \underset{\zeta}{=} \pm \frac{\# K_{4k-2}(\mathcal{O}_K)}{\# K_{4k-1}(\mathcal{O}_K)}$$

up to factors of 2.

2) $p \mid \# K_{4k}(\mathbb{Z})$ for some $k \geq 1$ $\Longleftrightarrow$

$p \mid \# \operatorname{Pic}(\mathbb{Z}(\zeta_p + \zeta_p^{-1}))$. //

Excision:

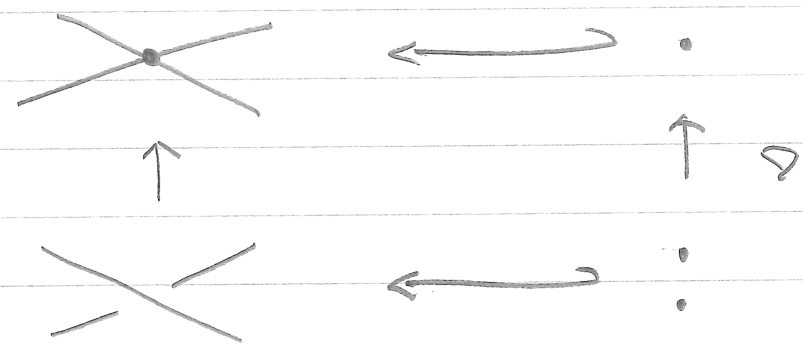
$$\begin{tikzcd} I \arrow[r] \arrow[d, "f", "\sim"'] & A \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\text{Cart.}"] & A/I \arrow[d, "\bar{f}"] \\ f(I) \arrow[r] & B \arrow[r] & B/f(I) \end{tikzcd} \overset{?}{\Longrightarrow}$$

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Cart.}"] & K(A/I) \arrow[d] \\ K(B) \arrow[r] & K(B/f(I)) \end{tikzcd}$$

Ex 1) k field

$$\begin{tikzcd} (x,y) \arrow[r] \arrow[d, equal] & k[x,y]/(xy) \arrow[r] \arrow[d] & k \arrow[d, "\Delta"] \\ (x,y) \arrow[r] & k[x] \times k[y] \arrow[r] & k \times k \end{tikzcd}$$

Spec $\longmapsto$



2) G : finite group

$$\begin{tikzcd}
|G|\cdot\mathfrak{m} \arrow[r] \arrow[d, equal] & \mathbb{Z}[G] \arrow[r] \arrow[d, hook] & \mathbb{Z}[G]/|G|\cdot\mathfrak{m} \arrow[d] \\
|G|\cdot\mathfrak{m} \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{m}/|G|\cdot\mathfrak{m}
\end{tikzcd}$$

$\mathfrak{m}$ $\leq$ maximal order in $\mathbb{Q}[G]$ //

Thm (Suslin – Wodzicki)

I is excisive for algebraic K-theory $\iff$

$$\mathrm{Tor}_q^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) = 0 \quad (q \geq 1).$$ //

Recall trace from linear algebra:

$$\begin{tikzcd}
\mathrm{Hom}_A(P,P) \arrow[drr, "\mathrm{tr}"'] & & P\otimes_A \mathrm{Hom}_A(P,A) \arrow[ll, "\sim"'] \arrow[d] & x\otimes f \arrow[d, mapsto] \\
& & A/[A,A] & f(x)
\end{tikzcd}$$

Hattori - Stallings trace map

$$K_0(A) \xrightarrow{tr} HH_0(A) = A/[A,A]$$

$$P \longmapsto tr(id_P)$$

Cyclotomic trace map

$$K(A) \xrightarrow{tr} TC(A),$$

very powerful refinement due to Connes, Bökstedt - Hsiang - Madsen.

Thm (Cortiñas, Geisser - H.)
Every I is excisive for the mapping fiber of the cyclotomic trace map. //

Ex 1) k: perfect field of pos. char.

$$\begin{tikzcd} I = (x,y) \arrow[r] \arrow[d, equal] & A = k[x,y]/(xy) \arrow[d] \\ I = (x,y) \arrow[r] & B = k[x] \times k[y] \end{tikzcd}$$

$$K_q(A,B,I) \xleftarrow{\sim} \begin{cases} \mathbb{W}_m(k) & q = 2m \text{ even} \\ 0 & q \text{ odd} \end{cases}$$

2) C_p = cyclic group of prime order p.

$$\begin{tikzcd}
I = (1-t) \arrow[r] \arrow[d, "\sim"] & A = \mathbb{Z}[C_p] \arrow[r] \arrow[d] & A/I = \mathbb{Z} \arrow[d] \\
f(I) = (1-\zeta_p) \arrow[r] & B = \mathbb{Z}(\zeta_p) \arrow[r] & B/f(I) = \mathbb{F}_p
\end{tikzcd}$$

Know: $K_q(A, B, I)$ is p-primary torsion group

Missing: $TC(\mathbb{Z}[C_p]) = ?$ "

Planar cusps :

$a < b$: relatively prime integers, $a > 1$.

$$\begin{array}{ccc} I = (x^i y^j) & \longrightarrow & A = k[x,y]/(y^a - x^b) \\ \downarrow \wr & & \downarrow f \\ f(I) = (t^{(a-1)(b-1)}) & \longrightarrow & B = k[t] \end{array}$$

$f(x) = t^a$, $f(y) = t^b$

$(a-1)(b-1) = ai + bj \qquad i, j \geq 0.$

Sylwester :

$$\text{length}_k(A/I) = \tfrac{1}{2}\,\text{length}_k(B/f(I)).$$

<u>Ex</u> $a = 3$, $b = 5$, $(a-1)(b-1) = a \cdot 1 + b \cdot 1$

$\underline{0} \quad 1 \quad 2 \quad \underline{3} \quad 4 \quad \underline{5} \quad \underline{6} \quad 7 \quad \underline{8} \quad \underline{9} \quad \underline{10} \quad \cdots$ //

To calculate $\kappa(A)$, must solve following geometric problem :

let $m \geq 1$ be an integer and let

$$p = (p_0, p_1, \ldots, p_r)$$

be integers $0 \leq p_0 < p_1 < \cdots < p_r \leq m$

s.t. the gaps

$$p_s - p_{s-1} \qquad (1 \le s \le r)$$

$$p_0 + m - p_r$$

all are equal to either a or b.
Choose integer c, d s.t.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Let

$$J = \left[\frac{cm}{a}, \frac{dm}{b} \right] \cap \mathbb{Z}$$

and consider the stunted trigonometric moment curve

$$[0, m] \xrightarrow{x} \mathbb{C}^J$$

$$x(t) = (x_j(t) \mid j \in J)$$

$$x_j(t) = \exp(2\pi i j t / m).$$

Define the polytope

$$\Sigma(a, b, m, p) \subset \mathbb{C}^J$$

to be the convex hull of

$\{x(p_0), \ldots, x(p_r)\} \subset \mathbb{C}^J$.

<u>Conj</u> $0 \notin \Sigma(a, b, m, p)$.

<u>Ex</u> $a = 2, \; b = 3, \; m = 7$

$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \in SL(2, \mathbb{Z})$

$J = \left[\frac{1 \cdot 7}{2}, \frac{2 \cdot 7}{3} \right] \cap \mathbb{Z} = \{4\}$

$p = (0, 2, 4)$