Survey on the classification of von Neumann factors of type II_1

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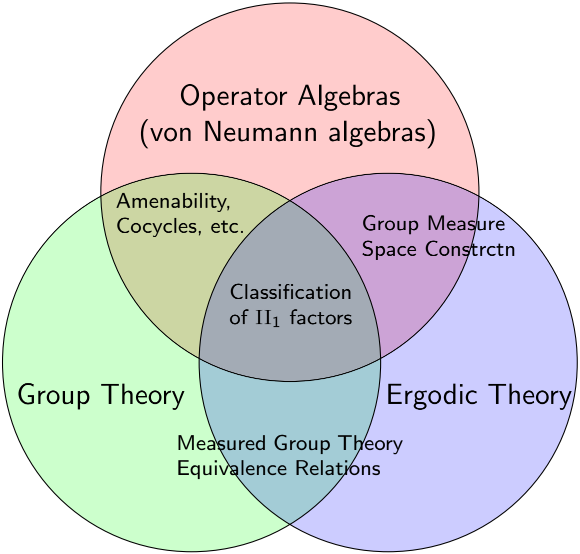
Overview of today's talk



Operator Algebras

$\mathbb{C}$ = the field of complex numbers (scalars)

$\mathbb{M}_n(\mathbb{C})$ = the $n \times n$ matrix algebra ("non commutative scalars")

$\mathbb{B}(\mathcal{H})$ = the $*$-algebra of bounded (continuous) linear operators on $\mathcal{H}$

$\mathcal{H}$ = a Hilbert space (mostly separable & infinite-dimensional)

$$\langle T\eta, \xi \rangle = \langle \eta, T^*\xi \rangle \quad \text{for } T \in \mathbb{B}(\mathcal{H}) \text{ and } \xi, \eta \in \mathcal{H}.$$

Definition

A $*$-subalgebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ is called

- a C^*-algebra if closed under the norm topology;
- a **von Neumann algebra** if closed under the weak-operator-topology.

Examples:

- $C_0(X) \subset \mathbb{B}(L^2(X,\mu))$ commutative C^*-algebras.
- $L^\infty(X,\mu) \subset \mathbb{B}(L^2(X,\mu))$ commutative vN algebras.
- The (reduced) group C^*-algebra $C^*_\lambda\Gamma \subset \mathbb{B}(\ell_2\Gamma)$ and the group vN algebra $\mathrm{vN}(\Gamma) \subset \mathbb{B}(\ell_2\Gamma)$.

Group von Neumann algebras

The group vN algebra:

$$\mathrm{vN}(\Gamma) := \text{WOT-closure of } \lambda(\mathbb{C}\Gamma) = \{\lambda(f) : \|\lambda(f)\| < \infty\} \subset \mathbb{B}(\ell_2\Gamma),$$

where $\lambda\colon \Gamma \curvearrowright \ell_2\Gamma, \quad \lambda_g\delta_x = \delta_{gx}; \qquad \lambda\colon \mathbb{C}\Gamma \to \mathbb{B}(\ell_2\Gamma), \quad \lambda(f)\xi = f * \xi.$

- Γ is abelian $\Longrightarrow \ell_2\Gamma \cong L^2(\widehat{\Gamma})$ and $\mathrm{vN}(\Gamma) \cong L^\infty(\widehat{\Gamma}) \cong L^\infty[0,1]$.
- $\mathrm{vN}(\Gamma)$ is a II_1-factor $\iff \mathbb{C}\Gamma$ has a trivial center
  $\iff \Gamma$ is ICC (Infinite Conjugacy Classes).

Examples of ICC groups: $\mathfrak{S}_\infty = \bigcup_n \mathfrak{S}_n,\ \mathbb{F}_r,\ \mathrm{PSL}(n,\mathbb{Z}), \ldots$.

Theorem (Murray–von Neumann 1943)

- $\mathrm{vN}(\Gamma)$ *are all isomorphic for countable locally finite ICC groups.*
- $\mathrm{vN}(\mathfrak{S}_\infty) \not\cong \mathrm{vN}(\mathbb{F}_r)$.

OPEN PROBLEM: ¿¿ $\mathrm{vN}(\mathbb{F}_r) \not\cong \mathrm{vN}(\mathbb{F}_s)$ for $r \neq s \in \{2, 3, \ldots, \infty\}$??

Classification Problem

geared for rigidity phenomena

Classification of (group) von Neumann algebras is very subtle. E.g.,

Theorem (Dykema 1993, Oz. 2006)

$\mathrm{vN}(\mathbb{F}_\infty * (\mathbb{F}_\infty \times \mathbb{Z})^{*n})$, $n = 1, 2, \ldots$, are mutually isomorphic, while $\mathrm{vN}(\mathbb{F}_\infty * (\mathbb{F}_\infty \times \mathfrak{S}_\infty)^{*n})$ are mutually non-isomorphic.

Moreover, Hjorth's theory of turbulence + Popa's rigidity theorem imply

Theorem (Sasyk–Törnquist 2009)

von Neumann algebras are not classifiable "by countable structures."

What do we classify?

Γ	countable discrete group
(X,μ)	standard probability measure space
$\Gamma \curvearrowright (X,\mu)$	(ergodic) measure preserving action

$\Gamma \curvearrowright X$ is *ergodic* if $A \subset X$ and $\Gamma A = A \Rightarrow \mu(A) = 0, 1$.
$\rightsquigarrow$ We consider only $(X,\mu) \cong ([0,1], \text{Lebesgue})$ or $X = \{\text{pt}\}$.
$\Gamma \curvearrowright X$ is *essentially-free* if $\mu(\{x : gx = x\}) = 0\ \forall g \in \Gamma \setminus \{1\}$.

Examples:

- $\Gamma = \mathbb{Z},\ \mathfrak{S}_\infty = \bigcup_n \mathfrak{S}_n,\ \mathbb{F}_r\ (2 \le r \le \infty),\ \mathrm{SL}(n,\mathbb{Z}), \ldots$
- $T\colon X \to X$ invertible p.m.p. transformation,
- $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$,
- $\Gamma \curvearrowright G/\Lambda$, $\Gamma, \Lambda \le G$ lattices,
- $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$, Bernoulli shift

Instead of X/Γ, we consider $X \rtimes \Gamma$.

Group measure space constrctn (Murray & vN '36 '43)

Instead of X/Γ, we consider $X \rtimes \Gamma$.

$\Gamma \curvearrowright (X,\mu)$ p.m.p. $\longleftrightarrow$ $\sigma\colon \Gamma \curvearrowright L^\infty(X,\mu)$, $\sigma_g(f)(x) = f(g^{-1}x)$, $\int \sigma_g(f)\,d\mu = \int f\,d\mu$

The unitary element $u_g = \sigma_g \otimes \lambda_g \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$ satisfies

$$u_g\, f\, u_g^* = \sigma_g(f)$$

for all $f \in L^\infty(X,\mu)$, identified with $f \otimes 1 \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$.
We encode the information of $\Gamma \curvearrowright X$ into a single vN algebra

$$\mathrm{vN}(X \rtimes \Gamma) := \{\sum_{g\in\Gamma}^{\text{finite}} f_g\, u_g : f_g \in L^\infty(X)\}'' \subset \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma)).$$

$\mathrm{vN}(X \rtimes \Gamma)$ is same as the crossed product vN algebra $L^\infty(X) \rtimes \Gamma$.

Group measure space constrctn (Murray & vN '36 '43)

$$\mathrm{vN}(X \rtimes \Gamma) = \{\sum_{g\in\Gamma} f_g\, u_g : f_g \in L^\infty(X)\}, \quad u_g\, f\, u_g^* = \sigma_g(f)$$

$\mathrm{vN}(X \rtimes \Gamma)$ is a vN algebra of type II_1, with the trace τ given by

$$\tau(\sum_g f_g\, u_g) = \langle \sum_g f_g\, u_g\, (\mathbf{1} \otimes \delta_1), (\mathbf{1} \otimes \delta_1)\rangle = \int f_1\, d\mu.$$

It follows $\tau(xy) = \tau(yx)$. $\rightsquigarrow$ a generalization of $(\mathbb{M}_n(\mathbb{C}),\ \frac{1}{n}\mathrm{Tr})$

The subalgebra $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ has a special property.

Definition

A von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates M as a von Neumann algebra.

$\rightsquigarrow$ Somewhat looks like a normal abelian subgroup.

Orbit Equivalence Relation

Theorem (Singer 1955, Dye, Krieger, Feldman–Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f\circ\theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

The orbit equivalence relation of $\Gamma \curvearrowright X$ is

$$\mathcal{R}_{\Gamma\curvearrowright X} := \{(x,y)\in X\times X : \exists g\in\Gamma \text{ s.t. } gx=y\} \subset X\times X,$$

a Borel equivalence relation with countable classes.

E.g., $(\Gamma \curvearrowright G/\Lambda) \cong_{\mathrm{OE}} (\Gamma\backslash G \curvearrowleft \Lambda)$ for lattices $\Gamma,\Lambda \le G$ of same covolume.

So, what is the classification problem?

GA	→	OE	→	vN
$\Gamma \curvearrowright (X,\mu)$		$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$		$\mathrm{vN}(X \rtimes \Gamma)$

GP	→	vN
Γ		$\mathrm{vN}(\Gamma)$

To what extent do vN/OE remember OE/GA/GP?

Amenable groups

Definition

A group Γ is *amenable* if $\exists$ a finitely additive measure m on 2^Γ which is translation invariant: $m(gS) = m(S)$ for $g \in \Gamma$ and $S \subset \Gamma$; or equivalently, if every action of Γ on a compact convex space has a fixed point.

Examples:

- finite and locally finite groups, e.g., $\mathfrak{S}_\infty = \bigcup_n \mathfrak{S}_n$,
- abelian groups and groups with subexponential growth,
- nilpotent and solvable groups,
- closed under subgroups, quotients, extensions and limits.

Non-example: Any group which contains the free group $\mathbb{F}_2 = \langle a, b \rangle$.

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup \tilde{B}^+ \sqcup \tilde{B}^-,$$

$$A^+ = \{a \cdots\},\ A^- = \{a^{-1} \cdots\},\ B^+ = \{b \cdots\},\ B^- = \{b^{-1} \cdots\};$$
$$\tilde{B}^+ = B^+ \setminus \{b, b^2, \ldots\},\ \tilde{B}^- = B^- \cup \{1, b, b^2, \ldots\}.$$

$\rightsquigarrow$ Banach–Tarski Paradox: $\mathbb{F}_2 = A^+ \sqcup a \cdot A^- = \tilde{B}^+ \sqcup b \cdot \tilde{B}^-$.

Lack of rigidity (**vN**)

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Hakeda–Tomiyama, Sakai 1967)

Γ *is amenable* $\Leftrightarrow$ $\mathrm{vN}(\Gamma)$ *and/or* $\mathrm{vN}(X \rtimes \Gamma)$ *is amenable (injective).*

Theorem (Connes 1974, Ornstein–Weiss, C–Feldman–W 1981)

Amenable **vN** *and* **OE** *are unique modulo center.*

Big Open Problem (Murray & von Neumann 1943)

$$\mathrm{vN}(\mathbb{F}_r) \not\cong \mathrm{vN}(\mathbb{F}_s) \qquad ?$$

They are either all isomorphic for $r = 2, 3, \ldots, \infty$, or all non-isomorphic (Voiculescu, Rădulescu, Dykema, around 1990).

Lack of rigidity (**OE**)

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Connes–Jones 1982)

OE $\longrightarrow$ **vN** *is not one-to-one,*
i.e. $\exists$ *a* II_1*-factor with non-conjugate Cartan subalgebras.*

Example (Oz–Popa 2008)

$$M = \mathrm{vN}(\mathbb{Z}_p^2 \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})))$$

has (at least) two Cartan subalgebras $L^\infty(\mathbb{Z}_p^2)$ and $\mathrm{vN}(\mathbb{Z}^2)$.

Speelman–Vaes 2011: $\exists$ a II_1-factor where classification of Cartan subalgebras is impossible.

Some rigidity phenomena

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	→	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	→	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Furman 99, Monod–Shalom, Popa, Kida, Popa–Vaes, . . .)

Some **OE** *fully remembers* **GA**.
E.g., $\mathrm{SL}(3,\mathbb{Z}) \curvearrowright \mathbb{T}^3$, $\Gamma \curvearrowright [0,1]^\Gamma$ *for many* Γ, $\mathrm{MCG}(\Sigma) \curvearrowright (X,\mu)$, . . .

Theorem (Oz–Popa 2007, Chifan–Sinclair 2011, Popa–Vaes 2011-12)

Some **vN** *fully remembers* **OE**, *i.e.,* $\exists$ *a (non-amenable)* II_1*-factor with a unique Cartan subalgebra (up to unitary conjugacy).*
In fact, every action of a free (hyperbolic) group is such an example.

Theorem (Popa–Vaes 2009, Ioana 2010, Chifan–Peterson 2010, . . .)

Some **vN** *fully remembers* **GA**. *E.g.*, $\Gamma \curvearrowright [0,1]^\Gamma$ *for ICC* $+$ (T) *groups* Γ.

Go back from **OE** to **GA**

Given an orbit equivalence relation

$$\mathcal{R}_{\Gamma \curvearrowright X} = \{(x,y) \in X \times X : \exists g \in \Gamma \text{ s.t. } gx = y\} \subset X \times X,$$

find Γ and $\Gamma \curvearrowright X$.

From OE to Cocycle (after Zimmer)

Suppose $(\Gamma \curvearrowright X) \cong_{\mathrm{OE}} (\Lambda \curvearrowright Y)$, i.e. $\exists\, \theta\colon X \xrightarrow{\sim} Y$ such that

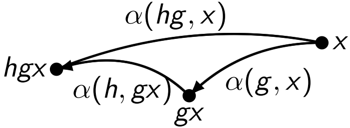
$$\theta(\Gamma x) = \Lambda\theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

Define $\alpha\colon \Gamma \times X \to \Lambda$ by

$$\theta(gx) = \alpha(g,x)\theta(x).$$

Then, α satisfies the cocycle identity:

$$\alpha(h, gx)\alpha(g,x) = \alpha(hg,x).$$



A cocycle α is a *homomorphism* if ess. independent of the second variable.
Cocycles α and β are *equivalent* if $\exists\, \phi\colon X \to \Lambda$ such that

$$\beta(g,x) = \phi(gx)\alpha(g,x)\phi(x)^{-1}.$$

Theorem (Zimmer)

$(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ *if and only if* α *is equivalent to a homomorphism.*

From Cocycle to Group Action

Theorem (Cocycle Superrigidity)

With some assumption on $\Gamma \curvearrowright X$ (and not on Λ), any cocycle

$$\alpha\colon \Gamma \times X \to \Lambda$$

is equivalent to a homomorphism β.

Applied to the Zimmer cocycle, one obtains (virtual) isomorphism $(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ via the homomorphism $\beta\colon \Gamma \to \Lambda$.

Examples

- Γ higher rank lattice + Λ simple Lie group (Zimmer 1981)
- Γ Kazhdan (T) / product + $\Gamma \curvearrowright X$ Bernoulli (Popa 2005-06)
- Γ Kazhdan (T) + $\Gamma \curvearrowright X$ profinite (Ioana 2008)
- $\Gamma \leq \mathrm{SL}(n \geq 5, \mathbb{R})$ and $\Gamma \curvearrowright \mathbb{R}^n$ (Popa–Vaes 2008)

$\rightsquigarrow$ Many applications to measured group theory & descriptive set theory.

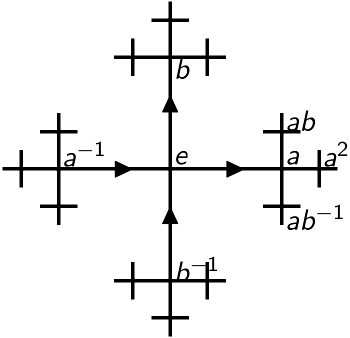
Go back from **vN** to **OE**

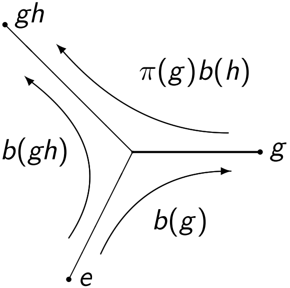
Given a group measure space von Neumann algebra $\mathrm{vN}(X \rtimes \Gamma)$, locate the position of the Cartan subalgebra $L^\infty(X)$ in $\mathrm{vN}(X \rtimes \Gamma)$.

The Cayley graph of $\mathbb{F}_2 = \langle a, a^{-1}, b, b^{-1} \rangle$ is an oriented tree.
$\mathbb{F}_2$ acts on the edge set E from the left: $\pi \colon \Gamma \curvearrowright \ell_2 E$,

$$b(g) := \text{signed char fnctn on the edge path } [e, g] \in \ell_2 E.$$

Then, $\|b(g)\|^2 = |g|$ and b satisfies the *cocycle condition*.





$$b(gh) = b(g) + \pi(g)b(h)$$

It follows that $\|b(g) - b(h)\|^2 = \|\pi(g)\big(b(g^{-1}h)\big)\|^2 = |g^{-1}h|$.

Noncommutative harmonic analysis

Theorem (Haagerup 1979)

$\phi_t(g) = \exp(-t|g|)$ *are positive definite on* $\mathbb{F}_r$ *and the multipliers*

$$m_{\phi_t}\colon \mathrm{vN}(\mathbb{F}_r) \ni \lambda(f) \mapsto \lambda(\phi_t f) \in \mathrm{vN}(\mathbb{F}_r)$$

are completely positive contractive maps which converge to id *as* $t \searrow 0$.

Moreover, ϕ_t can be perturbed to a sequence ψ_n of **finitely supported** functions on $\mathbb{F}_r$ such that $\psi_n \to 1$ and $\limsup \|m_{\psi_n}\|_{\mathrm{cb}} = 1$.

Compare this with Fejér's Theorem:

$\phi_n(k) = (1 - \frac{|k|}{n}) \vee 0$ are positive definite on $\mathbb{Z}$ and

$$m_{\phi_k}\colon C(\mathbb{T}) \ni h \sim \sum_k a_k z^k \mapsto \sum_k \phi_n(k) a_k z^k$$

are positive contractive maps which converge to id.

A group Γ is *weakly amenable* if it satisfies a similar property as above.

Ex.: Rank **one** Lie group lattices (..., Cowling–Haagerup, ...80s),
Hyperbolic groups (Oz. 08),
F.d. CAT(0) cube cplx (Niblo–Reeves, Guentner–Higson & Mizuta 07).

Rigidity results for Cartan subalgebras

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Voiculescu 94, OP 07, Chifan–Sinclair 11, PV 11-12)

Let Γ be an ICC non-amenable free group, hyperbolic group, or CAT(0) cube cplx grp whose hyperplane stabilizers are all amenable, etc.; and $\Gamma \curvearrowright (X,\mu)$ be any p.m.p. ergodic ess.-free action.

- $\mathrm{vN}(\Gamma)$ *does not have a Cartan subalgebra.*
  Hence $\mathrm{vN}(\Gamma) \not\cong \mathrm{vN}(Y \rtimes \Lambda)$ *for any* $\Lambda \curvearrowright (Y,\nu)$.
- $L^\infty(X)$ *is the unique Cartan subalgebra of* $\mathrm{vN}(X \rtimes \Gamma)$.

Combined with Gaboriau's theory of cost, this yields

$$\mathrm{vN}(X \rtimes \mathbb{F}_r) \not\cong \mathrm{vN}(Y \rtimes \mathbb{F}_s) \text{ for any } r \neq s.$$

OPEN PROBLEM: ¿¿ $\mathrm{vN}(\mathbb{F}_r) \not\cong \mathrm{vN}(\mathbb{F}_s)$ for $r \neq s \in \{2,3,\dots,\infty\}$??